

W -entropy formulas and Langevin deformation of flows on Wasserstein space over Riemannian manifolds

Songzi Li*, Xiang-Dong Li†

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Abstract

We introduce Perelman's W -entropy and prove the W -entropy formula along geodesic flow on the Wasserstein space $P_2^\infty(M, \mu)$ over compact Riemannian manifolds equipped with Otto's infinite dimensional Riemannian metric. As a corollary, we recapture Lott and Villani's result on the displacement convexity of $t\text{Ent} + mt \log t$ on $P_2^\infty(M, \mu)$ over Riemannian manifolds with Bakry-Emery's curvature-dimension $CD(0, m)$ -condition. To better understand the similarity between the W -entropy formula for the geodesic flow on the Wasserstein space and the W -entropy formula for the heat equation of the Witten Laplacian on the underlying manifolds, we introduce the Langevin deformation of flows on the Wasserstein space, which interpolates the geodesic flow and the gradient flow of the Boltzmann-Shannon entropy on the Wasserstein space over Riemannian manifolds, and can be regarded as the potential flow of the compressible Euler equation with damping on manifolds. We prove the local and global existence, uniqueness and regularity of the potential flow on compact Riemannian manifolds, and prove an analogue of the Perelman type W -entropy formula along the Langevin deformation of flows on the Wasserstein space on compact Riemannian manifolds. We also prove a rigidity theorem for the W -entropy for the geodesic flow and provide the rigidity models for the W -entropy for the Langevin deformation of flows on the Wasserstein space over complete Riemannian manifolds with the $CD(0, m)$ -condition. Finally, we prove the W -entropy inequalities along the geodesic flow, gradient flow and the Langevin deformation of flows on the Wasserstein space over compact Riemannian manifolds with Erbar-Kuwada-Sturm's entropic curvature-dimension $CD_{\text{Ent}}(K, N)$ -condition.

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1 Introduction

Entropy is an important tool in the study of evolutionary processes and can be used to characterize the equilibrium state of evolutionary equations. In 1872, L. Boltzmann [5] introduced the H-entropy $H = - \int f \log f dx dv$ for the probability distribution function $f = f(x, v)$ of particles moving in the phase space $T^*\mathbb{R}^3$, and formally derived the famous H-theorem for the Boltzmann equation and derived that the equilibrium of the H-entropy is given by the local Maxwell distribution. In 1948, Shannon introduced the Shannon entropy in his study of communication theory. In [25], J. Nash used the Boltzmann entropy to study the continuity property of the fundamental solution of the parabolic equation of uniformly elliptic

operators. In 2002, G. Perelman [29] introduced the \mathcal{F} -entropy and the \mathcal{W} -entropy for the Ricci flow and proved their monotonicity along the conjugate heat equation. As an application, Perelman proved the no local collapsing theorem which “removes the major stumbling block in Hamilton’s approach to geometrization”.

On the other hand, starting from Brenier’s work [6, 7] on the Monge-Kantorovich optimal transport problem with quadratic cost function, Otto, Lott, McCann, Villani and Sturm [24, 27, 19, 18, 37, 38, 31, 32, 33] among others have developed the optimal transport theory. In particular, they developed an infinite dimensional Riemannian geometry and the theory of the gradient flow on the Wasserstein space over Euclidean space, compact Riemannian manifolds and metric measure spaces. The convexity of the Boltzmann-Shannon entropy or the Renyi entropy along geodesics on the Wasserstein space has been a key tool in [19, 18, 37, 38, 31, 32, 33] to introduce the notions of the upper bound of the dimension and the lower bound of the Ricci curvature on metric measure spaces. In [22], McCann and Topping proved the contraction property of the L^2 -Wasserstein distance between solutions of the backward heat equation on closed manifolds equipped with the Ricci flow, which extends a previous result for the Fokker-Planck equation on Euclidean space (due to Otto [24]) and on complete Riemannian manifolds with suitable Bakry-Emery curvature condition (due to Sturm and von Renesse [34]). See also [35, 36]. In [18], Lott further proved two convexity results of the Boltzmann-Shannon type entropy along the geodesics on the Wasserstein space over closed manifolds equipped with Ricci flow, which are closely related to Perelman’s results on the monotonicity of the \mathcal{F} and \mathcal{W} -entropy functionals for Ricci flow. In [15], the authors extended Lott’s convexity results to the Wasserstein space on compact Riemannian manifolds equipped with Perelman’s Ricci flow.

Let (M, g) be a complete Riemannian manifold equipped with a weighted volume measure $d\mu = e^{-f} dv$, where $f \in C^2(M)$ and dv denotes the volume measure on (M, g) . The Boltzmann-Shannon entropy of the probability measure $\rho d\mu$ with respect to the reference measure μ is defined by

$$\text{Ent}(\rho) := \int_M \rho \log \rho d\mu.$$

Let $P_2(M, \mu)$ (resp. $P_2^\infty(M, \mu)$) be the Wasserstein space (reps. the smooth Wasserstein space) of all probability measures $\rho(x)d\mu(x)$ with density function (resp. with smooth density function) ρ on M such that $\int_M d^2(o, x)\rho(x)d\mu(x) < \infty$, where $d(o, \cdot)$ denotes the distance function from a fixed point $o \in M$. By Otto [24], the tangent space $T_{\rho d\mu} P_2^\infty(M, \mu)$ is identified as follows

$$T_{\rho d\mu} P_2^\infty(M, \mu) = \{s = -\nabla_\mu^*(\rho \nabla \phi) : \phi \in C^\infty(M), \int_M |\nabla \phi|^2 \rho d\mu < \infty\},$$

where ∇_μ^* denotes the L^2 -adjoint of the Riemannian gradient ∇ with respect to the weighted volume measure $d\mu$ on (M, g) . For $s_i = -\nabla_\mu^*(\rho \nabla \phi_i) \in T_{\rho d\mu} P_2^\infty(M, \mu)$, Otto [24] introduced the following infinite dimensional Riemannian metric on $P_2^\infty(M, \mu)$

$$\langle s_1, s_2 \rangle := \int_M \nabla \phi_1 \cdot \nabla \phi_2 \rho d\mu,$$

provided that

$$\|s_i\|^2 := \int_M |\nabla \phi_i|^2 \rho d\mu < \infty, \quad i = 1, 2.$$

Let $T_{\rho d\mu}P_2(M, \mu)$ be the completion of $T_{\rho d\mu}P_2^\infty(M, \mu)$ with Otto's Riemannian metric. Then $P_2(M, \mu)$ is an infinite dimensional Riemannian manifold.

By Benamou and Brenier [7], for any given $\mu_i = \rho_i d\mu \in P_2(M, \mu)$, $i = 0, 1$, the L^2 -Wasserstein distance between μ_0 and μ_1 coincides with the geodesic distance between μ_0 and μ_1 in $P_2(M, \mu)$ equipped with Otto's infinite dimensional Riemannian metric, i.e.,

$$W_2^2(\mu_0, \mu_1) = \inf \left\{ \frac{1}{2} \int_0^1 |\nabla \phi(x, t)|^2 \rho(x, t) d\mu(x) : \partial_t \rho + \nabla_\mu^*(\rho \nabla \phi) = 0, \rho(0) = \rho_0, \rho(1) = \rho_1 \right\}.$$

Given $\mu_0 = \rho(\cdot, 0)\mu$, $\mu_1 = \rho(\cdot, 1)\mu \in P_2^\infty(M, \mu)$, it is known that there is a unique minimizing Wasserstein geodesic $\{\mu(t), t \in [0, 1]\}$ of the form $\mu(t) = (F_t)_* \mu_0$ joining μ_0 and μ_1 in $P_2(M, \mu)$, where $F_t \in \text{Diff}(M)$ is given by $F_t(x) = \exp_x(-t \nabla \phi(\cdot, 0))$ for an appropriate Lipschitz function $\phi(\cdot, t)$ (see [21]). If the Wasserstein geodesic in $P_2(M, \mu)$ belongs entirely to $P_2^\infty(M, \mu)$, then the geodesic flow $(\rho, \phi) \in T^*P_2^\infty(M, \mu)$ satisfies the transport equation and the Hamilton-Jacobi equation

$$\partial_t \rho + \nabla_\mu^*(\rho \nabla \phi) = 0, \quad (1)$$

$$\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 = 0, \quad (2)$$

with the boundary condition $\rho(0) = \rho_0$ and $\rho(1) = \rho_1$. When $\rho_0, \phi_0 \in C^\infty(M)$, defining $\phi(\cdot, t) \in C^\infty(M)$ by the Hopf-Lax solution

$$\phi(x, t) = \inf_{y \in M} \left(\phi_0(y) + \frac{d^2(x, y)}{2t} \right), \quad (3)$$

and solving the transport equation (1) by the characteristic method, it is known that (ρ, ϕ) satisfies (1) and (2) with $\rho(0) = \rho_0$ and $\phi(0) = \phi_0$. See [37] Sect. 5.4.7. See also [17, 18]. In view of this, the transport equation (1) and the Hamilton-Jacobi equation (2) describe the geodesic flow on the cotangent bundle $T^*P_2^\infty(M, \mu)$ over the Wasserstein space $P_2(M, \mu)$. Note that the Hamilton-Jacobi equation (2) is also called the eikonal equation in geometric optics, see e.g. [8].

Our first result of this paper is the following W -entropy formula for the geodesic flow on the Wasserstein space $P_2(M, \mu)$.

Theorem 1.1 *Let (M, g) be a compact Riemannian manifold, $f \in C^2(M)$, $d\mu = e^{-f} dv$. Let $\rho : M \times [0, T] \rightarrow \mathbb{R}^+$ and $\phi : M \times [0, T] \rightarrow \mathbb{R}$ be smooth solutions to the transport equation (1) and the Hamilton-Jacobi equation (2). For any $m \geq n$, define the H_m -entropy and W_m -entropy for the geodesic flow (ρ, ϕ) on $T^*P_2^\infty(M, \mu)$ as follows*

$$H_m(\rho, t) = \text{Ent}(\rho(t)) + \frac{m}{2} (1 + \log(4\pi t^2)),$$

and

$$W_m(\rho, t) = \frac{d}{dt}(tH_m(\rho, t)).$$

Then for all $t > 0$, we have

$$\begin{aligned} \frac{d}{dt} W_m(\rho, t) &= t \int_M \left[\left| \text{Hess} \phi - \frac{g}{t} \right|^2 + \text{Ric}_{m,n}(L)(\nabla \phi, \nabla \phi) \right] \rho d\mu \\ &\quad + \frac{t}{m-n} \int_M \left| \nabla \phi \cdot \nabla f + \frac{m-n}{t} \right|^2 \rho d\mu. \end{aligned} \quad (4)$$

In particular, if $Ric_{m,n}(L) \geq 0$, then $W_m(\rho, t)$ is non-decreasing in time t along the geodesic flow on $T^*P_2^\infty(M, \mu)$.

Here and throughout this paper, for any constant $m \geq n$,

$$Ric_{m,n}(L) := Ric + \nabla^2 f - \frac{\nabla f \otimes \nabla f}{m-n}$$

is the m -dimensional Bakry-Emery Ricci curvature associated with the Witten Laplacian $L = \Delta - \nabla f \cdot \nabla$, and we use the convention that $m = n$ if and only if ϕ is identically a constant. When $m > n$ is an integer, $Ric_{m,n}(L)$ is the horizontal projection of the Ricci curvature on the product manifold $\widetilde{M} = M \times N$ with the warped product metric $\widetilde{g} = g \otimes e^{-\frac{2f}{m-n}} g_N$, where (N, g_N) is any $(m-n)$ -dimensional complete Riemannian manifold. Following [2], we say that the weighted Riemannian manifold (M, g, μ) satisfies the $CD(K, m)$ -condition for some constant $K \in \mathbb{R}$ and $m \in [n, \infty]$ if and only if $Ric_{m,n}(L) \geq K$.

As a corollary of Theorem 1.1, we can recapture the following beautiful result due to Lott-Villani [19, 18].

Corollary 1.2 ([19, 18]) *Let M be a compact Riemannian manifold. Suppose that $Ric_{m,n}(L) \geq 0$. Then $t\text{Ent}(\rho(t)) + mt \log t$ is convex in time t along the geodesic on $P_2(M, \mu)$.*

Recall that, in our previous papers [11, 13, 14], inspired by the work of Perelman [29] and Ni [26], we have proved the following

Theorem 1.3¹ [11, 13, 14] *Let M be a compact Riemannian manifold. Let u be a positive solution of the heat equation*

$$\partial_t u = Lu.$$

Define the H_m -entropy and the W_m -entropy as follows

$$H_m(u, t) = \text{Ent}(u(t)) + \frac{m}{2}(1 + \log(4\pi t)),$$

and

$$W_m(u, t) = \frac{d}{dt}(tH_m(u, t)).$$

Then

$$\begin{aligned} \frac{d}{dt}W_m(u, t) &= 2t \int_M \left[\left| \text{Hess} \log u + \frac{g}{2t} \right|^2 + Ric_{m,n}(L)(\nabla \log u, \nabla \log u) \right] u d\mu \\ &\quad + \frac{2t}{m-n} \int_M \left| \nabla \log u \cdot \nabla f - \frac{m-n}{2t} \right|^2 u d\mu. \end{aligned} \quad (5)$$

In particular, if $Ric_{m,n}(L) \geq 0$, then $W_m(u, t)$ is non-decreasing in time t along the heat equation $\partial_t u = Lu$.

¹The W -entropy formula (5) can be regarded as an analogue of Perelman's W -entropy formula for the Ricci flow, and extends Ni's W -entropy formula for the heat equation of the Laplace-Beltrami operator on Riemannian manifolds with non-negative Ricci curvature [26]. In [14], when $m \in \mathbb{N}$, we gave an alternative proof of the W -entropy formula (5) by applying Ni's result to $\widetilde{M} = M \times N$ equipped with the warped product metric $\widetilde{g} = g \otimes e^{-\frac{2f}{m-n}} g_N$, where (N, g_N) is any $(m-n)$ dimensional compact Riemannian manifold. We also extended the W -entropy formula (5) to the heat equation of the time dependent Witten Laplacian on compact Riemannian manifolds equipped with time dependent metrics and potentials, and to Witten Laplacian on complete Riemannian manifolds with the $CD(K, m)$ -condition, see [14, 16].

Similarly to Corollary 1.2, as a corollary of Theorem 1.3, we have the following

Corollary 1.4 *Let M be a compact Riemannian manifold. Suppose that $\text{Ric}_{m,n}(L) \geq 0$. Then $t\text{Ent}(u(t)) + \frac{m}{2}t \log t$ is convex in time t along the heat equation $\partial_t u = Lu$ on M .*

We now want to compare the W -entropy formula (4) in Theorem 1.1 and the W -entropy formula (5) in Theorem 1.3. They have some similar but also some different features. Indeed, Theorem 1.3 can be extended to complete Riemannian manifolds with bounded geometric condition (see [13]) and a rigidity theorem has been also proved on complete Riemannian manifolds with $CD(0, m)$ -condition: $W_m(u, t)$ achieves its minimum at some $t = t_0 > 0$ if and only if $M = \mathbb{R}^n$, $m = n$, and $u(x, t) = u_m(x, t) = \frac{1}{(4\pi t)^{\frac{m}{2}}} e^{-\frac{\|x\|^2}{4t}}$ is the heat kernel of the heat equation $\partial_t u = \Delta u$ on \mathbb{R}^m . Note that, the Boltzmann-Shannon entropy of the Gaussian heat kernel measure $u_m(x, t)dx$ is given by

$$\text{Ent}(u_m(t)) = -\frac{m}{2}(1 + \log(4\pi t)).$$

Thus the H_m -entropy for the heat equation of the Witten Laplacian is indeed given by²

$$H_m(u(t)) = \text{Ent}(u(t)) - \text{Ent}(u_m(t)),$$

and the W_m -entropy for the heat equation of the Witten Laplacian is given by the Boltzmann entropy formula

$$W_m(u, t) := \frac{d}{dt} (t[\text{Ent}(u(t)) - \text{Ent}(u_m(t))]). \quad (6)$$

This gives a natural probabilistic interpretation of the W -entropy for the heat equation of the Witten Laplacian on Riemannian manifolds. See also [13] for the probabilistic interpretation of the Perelman W -entropy for the Ricci flow.

On the other hand, when $m \in \mathbb{N}$, we can easily check that the following (ρ_m, ϕ_m)

$$\begin{aligned} \phi_m(x, t) &= \frac{\|x\|^2}{2t}, \\ \rho_m(x, t) &= \frac{1}{(4\pi t^2)^{m/2}} e^{-\frac{\|x\|^2}{4t^2}}, \end{aligned}$$

where $t > 0, x \in \mathbb{R}^m$, is a solution to the transport equation (1) and the Hamilton-Jacobi equation (2) on \mathbb{R}^m equipped with the standard Lebesgue measure, i.e.,

$$\partial_t \rho + \text{div}(\rho \nabla \phi) = 0, \quad (7)$$

$$\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 = 0, \quad (8)$$

Moreover, the Boltzmann-Shannon entropy of the probability measure $\rho_m(t, x)dx$ (which equals to $u_m(t^2, x)dx$) is given by

$$\text{Ent}(\rho_m(t)) = -\frac{m}{2}(1 + \log(4\pi t^2)).$$

²Following Villani [37, 38], we call $H_m(u(t))$ the *relative entropy* even though it is slightly different from the classical definition of the relative entropy in probability theory.

Thus we can reformulate the H_m -entropy and the W_m -entropy for the geodesic flow on the Wasserstein space $P_2(M, \mu)$ as follows

$$H_m(\rho(t)) = \text{Ent}(\rho(t)) - \text{Ent}(\rho_m(t)), \quad (9)$$

and

$$W_m(\rho, t) := \frac{d}{dt} (t[\text{Ent}(\rho(t)) - \text{Ent}(\rho_m(t))]). \quad (10)$$

Note that $H_m(\rho(t))$ defined by (9) is the difference between the Boltzmann-Shannon entropy of the probability measure $\rho(t)d\mu$ on (M, μ) and the Boltzmann-Shannon entropy of the reference model $\rho_m(t)dx$ on (\mathbb{R}^m, dx) , and $W_m(\rho, t)$ defined by (10) is the time derivative of $tH_m(\rho(t))$. Indeed, similarly to the case of Theorem 1.3, we can extend the W -entropy formula (4) in Theorem 1.1 to complete Riemannian manifolds. To do so, we need overcome some technical issue to verify the entropy dissipation formulas for smooth solutions of the transport equation and the Hamilton-Jacobi equation on complete Riemannian manifolds. In view of this, we can prove that the rigidity model for the W -entropy for the geodesic flow on the Wasserstein space $P_2^\infty(M, \mu)$ over complete Riemannian manifolds with the $CD(0, m)$ -condition, i.e., $\text{Ric}_{m,n}(L) \geq 0$, is the Euclidean space $M = \mathbb{R}^n$, $m = n$ and with $\rho = \rho_m$ and $\phi = \phi_m$. For detail, see Section 4.

We can raise a natural question how to understand the similarity between the W -entropy formulas in Theorem 1.1 and Theorem 1.3. Can we pass through one of them to another one? One possible approach to answer this question is to use the vanishing viscosity limit method from the heat equation to the Hamilton-Jacobi equation. However, it seems that one cannot easily use this approach to pass through the W -entropy formula for the heat equation of the Witten Laplacian to the W -entropy formula for the geodesic flow on the Wasserstein space. In this paper, inspired by J.-M. Bismut's work (see [3, 4]) on the deformation of hypoelliptic Laplacians on the cotangent bundle over Riemannian manifolds, which interpolate the usual Laplacian on the underlying Riemannian manifold M and the Hamiltonian vector field which generates the geodesic flow on the cotangent bundle over M , we introduce a deformation of geometric flows on the cotangent bundle of the Wasserstein space over compact Riemannian manifolds, and prove an analogue of the W -entropy formula along each deformed geometric flow on the cotangent bundle over the Wasserstein space.

We now describe how to introduce the deformation of geometric flows on $T^*P_2(M, \mu)$. For any $c \geq 0$, define the geometric flow $(\rho, \phi) : [0, T] \rightarrow T^*P_2^\infty(M, \mu)$ by solving the following equations on $T^*P_2^\infty(M, \mu)$

$$\partial_t \rho + \nabla_\mu^*(\rho \nabla \phi) = 0, \quad (11)$$

$$c^2 \left(\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 \right) = -\phi + \log \rho + 1, \quad (12)$$

where $T^*P_2^\infty(M, \mu)$ is the cotangent bundle over the Wasserstein space $P_2^\infty(M, \mu)$. The first equation is the transport equation with respect to the weighted volume measure μ , and the second equation, which we call the deformed Hamilton-Jacobi equation, can be regarded as the Newton-Langevin equation on the cotangent bundle over the Wasserstein space $P_2^\infty(M, \mu)$. Taking $u = \nabla \phi$, we can easily check that (ρ, u) is a solution to the following

compressible Euler equation with damping on (M, μ)

$$\partial_t \rho + \nabla_\mu^*(\rho u) = 0, \quad (13)$$

$$\partial_t u + u \cdot \nabla u = -\frac{u}{c^2} + \frac{1}{c^2} \frac{\nabla \rho}{\rho}. \quad (14)$$

In Section 6, using the Kato-Majda theory of the symmetric hyperbolic quasi-linear systems, we prove that, for any given $c > 0$, there exists $T = T_c > 0$ such that the Cauchy problem of the system (13) and (14) has a unique smooth solution $(\rho, u) \in C^1([0, T], C^\infty(M, \mathbb{R}^+ \setminus \{0\}) \times C(M))$ with given initial data $(\rho_0, u_0) \in C^\infty(M, \mathbb{R}^+ \setminus \{0\}) \times C^\infty(M)$, and the Cauchy problem of the system (11) and (12) has a unique smooth solution $(\rho, \phi) \in C^1([0, T], C^\infty(M, \mathbb{R}^+ \setminus \{0\}) \times C(M))$ with given initial data $(\rho_0, \phi_0) \in C^\infty(M, \mathbb{R}^+ \setminus \{0\}) \times C^\infty(M)$. Moreover, we can prove that, when the initial data (ρ_0, u_0) or (ρ_0, ϕ_0) is small enough in suitable Sobolev norm, $T_c = \infty$.

The limiting cases $c \rightarrow 0$ and $c \rightarrow \infty$ can be specified as follows. When $c = 0$, from (12) we have

$$\phi = \log \rho + 1 = \frac{\delta \text{Ent}(\rho)}{\delta \rho}, \quad (15)$$

which is the L^2 -derivative of the Boltzmann-Shannon entropy Ent on $P_2(M, \mu)$ equipped with Otto's infinite dimensional Riemannian metric ([24, 37, 38]). In this case, ρ satisfies the backward heat equation

$$\partial_t \rho = -L\rho, \quad (16)$$

Equivalently, when $c = 0$, (ρ, ϕ) can be regarded as the backward gradient flow of the Boltzmann-Shannon entropy on $P_2(M, \mu)$ equipped with Otto's infinite dimensional Riemannian metric. When $c \rightarrow \infty$, to make the sense of the equation (12), ρ and ϕ must satisfies the transport equation (1) and the Hamilton-Jacobi equation (2), i.e., (ρ, ϕ) is the geodesic flow on the cotangent bundle over the Wasserstein space $P_2(M, \mu)$. Thus, the family of flows $\{(\rho, \phi) : c \in [0, \infty]\}$ is a deformation of geometric flows which interpolate the backward gradient flow of the Boltzmann-Shannon entropy on $P_2(M, \mu)$ (i.e., the backward heat equation (16) of the Witten Laplacian on the underlying manifold (M, g, μ)) and the geodesic flow on the cotangent bundle over the Wasserstein space $P_2(M, \mu)$.

The following result will be proved in Section 5.

Theorem 1.5 *Let (M, g) be a compact Riemannian manifold, $f \in C^2(M)$, $d\mu = e^{-f} dv$. For any $c \geq 0$, let (ϕ, ρ) be a smooth solution to the transport equation (11) and the deformed Hamilton-Jacobi equation (12). Let*

$$H(\rho, \phi) = \frac{c^2}{2} \int_M |\nabla \phi|^2 \rho d\mu + \int_M \rho \log \rho d\mu.$$

Then

$$\frac{d^2}{dt^2} H(\rho, \phi) = 2 \int_M [c^{-2} |\nabla \phi - \nabla \log \rho|^2 + |\text{Hess} \phi|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi)] \rho d\mu.$$

In particular, if the $CD(0, \infty)$ -condition holds, i.e., $\text{Ric}(L) = \text{Ric} + \text{Hess} f \geq 0$, then $H(\rho, \phi)$ is convex along the deformed flow (ρ, ϕ) defined by (11) and (12).

To establish the W -entropy type formula for the deformed geometric flow on $T^*P_2(M, \mu)$, we need first to introduce the reference model in order to define the relative entropy functional as in (9). In Section 6, we prove that there is a special solution to the transport equation (11) and the deformed Hamilton-Jacobi equation (12) on Euclidean space (\mathbb{R}^m, dx) when $m \in \mathbb{N}$. More precisely, let $T > 0$, let $u : [0, T) \rightarrow (0, \infty)$ be a smooth solution to the ODE

$$c^2 u'' + u' = -\frac{1}{2u} \quad (17)$$

with given initial datas $u(0) > 0$ and $u'(0) \in \mathbb{R}$. Let $\alpha(t) = \frac{u'(t)}{u(t)}$, and let $\beta(t) \in C([0, T), \mathbb{R})$ such that

$$c^2 \dot{\beta}(t) = -\beta(t) - m \log u(t) - \frac{m}{2} \log(4\pi) + 1.$$

For $x \in \mathbb{R}^m$ and $t \in [0, T)$, let us introduce

$$\begin{aligned} \phi_m(x, t) &= \frac{\alpha(t)}{2} \|x\|^2 + \beta(t), \\ \rho_m(x, t) &= \frac{1}{(4\pi u^2(t))^{m/2}} e^{-\frac{\|x\|^2}{4u^2(t)}}. \end{aligned}$$

Then (ρ_m, ϕ_m) is a smooth solution of (11) and (12) on (\mathbb{R}^m, dx) , i.e.,

$$\begin{aligned} \partial_t \rho + \operatorname{div}(\rho \nabla \phi) &= 0, \\ c^2 \left(\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 \right) &= -\phi + \log \rho + 1. \end{aligned}$$

The above (ρ_m, ϕ_m) can be regarded as a rigidity model to (11) and (12). When $c = 0$, for any $T > 0$, we can take $u(t) = \sqrt{T-t}$, $\alpha(t) = -\frac{1}{2} \frac{1}{T-t}$ and $\beta(t) = -\frac{m}{2} \log(4\pi(T-t)) + 1$, $t \in [0, T)$. When $c = \infty$, for any $T > 0$, we can take $u(t) = T-t$, $\alpha(t) = -\frac{1}{T-t}$ and $\beta(t) = 0$, $t \in [0, T)$, or $u(t) = t$, $\alpha(t) = -\frac{1}{t}$ and $\beta(t) = 0$, $t \in (0, T]$.

The following result will be proved in Section 6, which can be viewed as a variant of the W -entropy formula for the Langevin deformation of geometric flow on $T^*P_2^\infty(M, \mu)$, and interpolate the W -entropy formula for the geodesic flow on $T^*P_2^\infty(M, \mu)$ and the backward gradient flow of the Boltzmann-Shannon entropy on $P_2(M, \mu)$.

Theorem 1.6 *Let (M, g) be a compact Riemannian manifold, $f \in C^2(M)$, $d\mu = e^{-f} dv$. For any $c \geq 0$, let (ϕ, ρ) be a smooth solution to the transport equation (11) and the deformed Hamilton-Jacobi equation (12). Then*

$$\begin{aligned} & \frac{d^2}{dt^2} \operatorname{Ent}(\rho(t)) + \left(2\alpha(t) + \frac{1}{c^2} \right) \frac{d}{dt} \operatorname{Ent}(\rho(t)) + m\alpha^2(t) \\ &= \int_M \left[|\operatorname{Hess} \phi - \alpha(t)g|^2 + \operatorname{Ric}_{m,n}(L)(\nabla \phi, \nabla \phi) \right] \rho d\mu \\ & \quad + (m-n) \int_M \left| \alpha(t) + \frac{\nabla \phi \cdot \nabla f}{m-n} \right|^2 \rho d\mu + \frac{1}{c^2} \int_M \frac{|\nabla \rho|^2}{\rho} d\mu. \end{aligned} \quad (18)$$

In particular, if the $CD(0, m)$ -condition holds, i.e., $Ric_{m,n}(L) \geq 0$, we have

$$\frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \left(2\alpha(t) + \frac{1}{c^2}\right) \frac{d}{dt} \text{Ent}(\rho(t)) + m\alpha^2(t) \geq \frac{1}{c^2} \int_M \frac{|\nabla \rho|^2}{\rho} d\mu.$$

In particular, when $m = n$ and $\mu = v$, we have the following result on compact Riemannian manifolds with standard volume measure.

Theorem 1.7 *Let (M, g) be a compact Riemannian manifold. For any $c \geq 0$, let (ϕ, ρ) be a smooth solution to the transport equation and the deformed Hamilton-Jacobi equation*

$$\partial_t \rho + \text{div}(\rho \nabla \phi) = 0, \quad (19)$$

$$c^2 \left(\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 \right) = -\phi + \log \rho + 1. \quad (20)$$

Then

$$\begin{aligned} & \frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \left(2\alpha(t) + \frac{1}{c^2}\right) \frac{d}{dt} \text{Ent}(\rho(t)) + n\alpha^2(t) \\ &= \int_M \left[|\text{Hess} \phi - \alpha(t)g|^2 + \text{Ric}(\nabla \phi, \nabla \phi) \right] \rho dv + \frac{1}{c^2} \int_M \frac{|\nabla \rho|^2}{\rho} dv. \end{aligned}$$

In particular, if $\text{Ric} \geq 0$, then along the deformed flow (ρ, ϕ) defined by (19) and (20), we have

$$\frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \left(2\alpha(t) + \frac{1}{c^2}\right) \frac{d}{dt} \text{Ent}(\rho(t)) + n\alpha^2(t) \geq \frac{1}{c^2} \int_M \frac{|\nabla \rho|^2}{\rho} dv.$$

To end this section, let us give some remarks and comments on Theorem 1.1, Theorem 1.3 and Theorem 1.6.

- In the case $c = \infty$, (ρ, ϕ) is the geodesic flow on $T^*P_2(M, \mu)$, $u(t) = t$, $\alpha(t) = \frac{1}{t}$ and

$$\begin{aligned} \phi_m(x, t) &= \frac{\|x\|^2}{2t}, \\ \rho_m(x, t) &= \frac{1}{(4\pi t^2)^{m/2}} e^{-\frac{\|x\|^2}{4t^2}}. \end{aligned}$$

In this case, we have

$$\frac{d^2}{dt^2} \text{Ent}(\rho(t)) = \int_M \left[|\text{Hess} \phi|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi) \right] \rho d\mu,$$

and

$$\begin{aligned} & \frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \frac{2}{t} \frac{d}{dt} \text{Ent}(\rho(t)) + \frac{m}{t^2} \\ &= \int_M \left[\left| \text{Hess} \phi - \frac{g}{t} \right|^2 + \text{Ric}_{m,n}(L)(\nabla \phi, \nabla \phi) \right] \rho d\mu \\ & \quad + \frac{1}{m-n} \int_M \left| \nabla \phi \cdot \nabla f + \frac{m-n}{t} \right|^2 \rho d\mu. \end{aligned} \quad (21)$$

This is an equivalent form of the W -entropy formula for the geodesic flow on $T^*P_2^\infty(M, \mu)$ in Theorem 1.1.

- In the case $c = 0$, $\phi = \log \rho + 1$, $\partial_t \rho = -L\rho$. For any $T > 0$, let $\tau = T - t$. Then $\partial_\tau \rho(\tau) = L\rho(\tau)$, $u(t) = \sqrt{T-t} = \sqrt{\tau}$ is a solution to (17). Thus $\alpha(t) = -\frac{1}{2} \frac{1}{T-t} = -\frac{1}{\tau}$, and

$$\begin{aligned}\phi_m(x, t) &= -\frac{\|x\|^2}{4\tau} - \frac{m}{2} \log(4\pi\tau) + 1, \\ \rho_m(x, t) &= \frac{1}{(4\pi\tau)^{m/2}} e^{-\frac{\|x\|^2}{4\tau}}.\end{aligned}$$

In this case, we have

$$\frac{d^2}{d\tau^2} \text{Ent}(\rho(\tau)) = 2 \int_M [\text{Hess} \log \rho]^2 + \text{Ric}(L)(\nabla \log \rho, \nabla \log \rho)] \rho d\mu,$$

and we can prove

$$\begin{aligned}& \frac{d^2}{d\tau^2} \text{Ent}(\rho(\tau)) + \frac{2}{\tau} \frac{d}{d\tau} \text{Ent}(\rho(\tau)) + \frac{m}{2\tau^2} \\ &= 2 \int_M \left[\left| \text{Hess} \log \rho + \frac{g}{2\tau} \right|^2 + \text{Ric}_{m,n}(L)(\nabla \log \rho, \nabla \log \rho) \right] \rho d\mu \\ & \quad + \frac{2}{m-n} \int_M \left| \nabla \log \rho \cdot \nabla f - \frac{m-n}{2\tau} \right|^2 \rho d\mu.\end{aligned}\tag{22}$$

This is an equivalent form of the W -entropy formula heat equation $\partial_\tau \rho = L\rho$ in Theorem 1.3.

- In general case $0 < c < \infty$, Theorem 1.6 suggests us to introduce a variant of the W -entropy as follows

$$W(\rho(t)) := \frac{d}{dt} \text{Ent}(\rho(t)) + 2 \int_0^t \alpha(s) \frac{d}{ds} \text{Ent}(\rho(s)) ds + \frac{1}{c^2} \text{Ent}(\rho(t)) - \frac{1}{c^2} \int_0^t \int_M \frac{|\nabla \rho|^2}{\rho} d\mu ds.$$

By direct calculation we can verify that

$$\frac{d}{dt} W(\rho_m(t)) = -m\alpha^2(t).$$

In view of this, Theorem 1.6 can be reformulated as follows

$$\begin{aligned}\frac{d}{dt} (W(\rho(t)) - W(\rho_m(t))) &= \int_M |\text{Hess} \phi - \alpha(t)g|^2 \rho d\mu + \int_M \text{Ric}_{m,n}(L)(\nabla \phi, \nabla \phi) \rho d\mu \\ & \quad + (m-n) \int_M \left| \alpha(t) + \frac{\nabla \phi \cdot \nabla f}{m-n} \right|^2 \rho d\mu.\end{aligned}$$

In particular, if $\text{Ric}_{m,n}(L) \geq 0$, then for all $t > 0$, we have the comparison theorem

$$\frac{d}{dt} W(\rho(t)) \geq \frac{d}{dt} W(\rho_m(t)).$$

- The entropy formulas in Theorem 1.1, Theorem 1.3 and Theorem 1.6 suggest that the special solution (ρ_m, ϕ_m) should play as the reference model for the rigidity theorem of the W -entropy on Riemannian manifolds with $CD(0, m)$ -condition. Indeed, in [12, 14], the W -entropy formula (5) has been extended to complete Riemannian manifolds with bounded geometry condition and the following rigidity theorem is proved: Let M be a complete Riemannian manifold with bounded geometry condition and with $CD(0, m)$ -condition, i.e., $Ric_{m,n}(L) \geq 0$. Then $\frac{d}{dt}W_m(u, t) = 0$ holds at some $t = t_0 > 0$ if and only if M is isometric to \mathbb{R}^n , $m = n$, f is a constant, $L = \Delta$, and $u = u_m$ is the standard Gaussian heat kernel on \mathbb{R}^n . Thus, the Gaussian heat kernel u_m on \mathbb{R}^m can be considered as the reference model for the W_m -entropy for the heat equation of the Witten Laplacian.

Similarly, for $c \in (0, \infty]$, we can expect to extend Theorem 1.1 and Theorem 1.6 to a class of smooth solutions (ρ, ϕ) on complete Riemannian manifolds with natural bounded geometry condition. We can therefore expect that the following rigidity theorem holds: Let M be a complete Riemannian manifold with natural bounded geometry condition and with $CD(0, m)$ -condition, i.e., $Ric_{m,n}(L) \geq 0$. Then $\frac{d}{dt}W_m(\rho, t) = 0$ or $\frac{d}{dt}W(\rho(t)) = \frac{d}{dt}W(\rho_m(t))$ holds at some $t = t_0 > 0$, if and only if M is isometric to \mathbb{R}^n , $m = n$, f is a constant, $(\rho, \phi) = (\rho_m, \phi_m)$. In view of this, (ρ_m, ϕ_m) can be considered as the reference model for the W_m -entropy for the optimal transport problem or the deformed optimal transport problem on complete Riemannian manifolds with natural bounded geometry condition and with $CD(0, m)$ -condition.

- Let $U_\mu(\rho) = \int_M U(\rho) d\mu$, where $U : [0, \infty) \rightarrow \mathbb{R}$ is a continuous convex function with $U(0) = 0$. By Remark 2 and Remark 3 in [18], if the $CD(0, m)$ -condition holds, then $tU_\mu(\rho(t)) + mt \log t$ is convex along the geodesics flow $\rho(t)d\mu$ on $P_2^\infty(M, \mu)$. Similarly to Theorem 1.1 and Theorem 1.3, we can introduce the W -entropy with respect to the Renyi entropy U_μ and prove the W -entropy formula for the geodesic flow and the gradient flow of $U_\mu(\rho) = \int_M U(\rho) d\mu$ on the Wasserstein space, i.e., the porous media equation $\partial_t \rho = \nabla_\mu^*(\rho \nabla U'(\rho))$. Moreover, we can also extend the W -entropy formula to the Langevin deformation of geometric flows which interpolates the porous media equation on (M, g, μ) and the geodesic flow on $P_2^\infty(M, \mu)$. Due to the limit of the length of the paper, we will develop this in a forthcoming paper.

In [19, 31, 32, 33], Lott-Villani and Sturm proved that the Boltzmann-Shannon entropy Ent is K -convex along the geodesic on the Wasserstein space $P_2(M, \mu)$ if and only if the $CD(K, \infty)$ -condition holds, i.e., $Ric(L) \geq K$, and the Rényi entropy $S_N(\rho\mu) = -\int_M \rho^{-1/N} d\mu$ is convex along the geodesic on $P_2(M, \mu)$ for all $N \geq m$ if and only if the $CD(0, m)$ -condition holds. In view of this, we would like to raise the following conjecture for the characterization of the $CD(0, m)$ -condition on complete Riemannian manifolds, which can be regarded as the converse of Corollary 1.2 and Corollary 1.4.

Conjecture 1.8 *Let (M, g) be a compact Riemannian manifold or a complete Riemannian manifolds with bounded geometry condition, $f \in C^\infty(M)$ with $\nabla f \in C_b^\infty(M)$. Suppose that the W_m -entropy associated to the heat equation of the Witten Laplacian or the optimal transport problem is non-decreasing in $t \in [0, T]$. Then the $CD(0, m)$ -condition holds, i.e., $Ric_{m,n}(L) \geq 0$.*

The rest of this paper is organized as follows. In Section 2 we calculate the Hessian of the Boltzmann entropy Ent on the Wasserstein space $P_2(M, \mu)$ on compact Riemannian

manifolds equipped with weighted volume measure. In Section 3 we give a new proof of the W -entropy formula (5) for the heat equation of the Witten Laplacian on Riemannian manifolds, i.e., Theorem 1.3. In Section 4 we introduce the W -entropy and prove the W -entropy formula for the geodesic flow on Wasserstein space, i.e., Theorem 1.1. Moreover, we extend Theorem 1.1 to complete Riemannian manifolds with bounded geometry condition and establish a rigidity theorem for the W -entropy for the geodesic flow on Wasserstein space. See Theorem 4.7. In Section 5 we introduce the Langevin deformation of geometric flows on the cotangent bundle of a complete Riemannian manifold and prove the variational formula for the Hamiltonian function. In Section 6 we introduce the Langevin deformation of geometric flows on the Wasserstein space over Riemannian manifolds. We prove the existence, uniqueness and regularity of the solution to the Cauchy problem of the compressible Euler equation with damping on compact Riemannian manifolds and the Cauchy problem of the Langevin deformation of geometric flows on $T^*P_2^\infty(M, \mu)$. In Section 7 we prove the entropy dissipation formula for the geometric flows on the Wasserstein space, i.e., Theorem 1.5. In Section 8 we prove Theorem 1.6. In Section 9 we prove the W -entropy inequality under the Erbar-Kawada-Sturm entropy curvature-dimension condition.

2 Otto's calculus on Wasserstein space over weighted Riemannian manifolds

Let (M, g) be a complete Riemannian manifold, $f \in C^2(M)$, and $d\mu = e^{-f} dv$, where dv denotes the volume measure on (M, g) . Integration by parts formula show that, for all $u \in C_0^\infty(M)$ and $X \in C_0^\infty(M, TM)$, we have

$$\int_M \langle X, \nabla u \rangle d\mu = \int_M \nabla_\mu^* X u d\mu,$$

where ∇_μ^* denotes the L^2 -adjoint of ∇ with respect to μ , and is given by

$$\nabla_\mu^* X = -\nabla \cdot X + \langle \nabla f, X \rangle.$$

The Witten Laplacian on (M, g) with respect to μ is defined by

$$L = -\nabla_\mu^* \nabla.$$

More precisely, we have

$$L = \Delta - \nabla f \cdot \nabla.$$

By Bakry-Emery [2], the Bochner-Weitzenböck formula holds

$$L|\nabla u|^2 - 2\langle \nabla u, \nabla Lu \rangle = 2|\text{Hess}u|^2 + 2\text{Ric}(L)(\nabla u, \nabla u), \quad \forall u \in C^\infty(M),$$

where

$$\text{Ric}(L) := \text{Ric} + \nabla^2 f$$

is the infinite dimensional Bakry-Emery Ricci curvature associated with the Witten Laplacian L . Following [2], we say that $CD(K, \infty)$ -condition holds if and only if $\text{Ric}(L) \geq K$.

Let

$$P^\infty(M, \mu) = \{\rho d\mu : \rho \in C^\infty(M), \rho \geq 0, \int_M \rho d\mu = 1\}.$$

For all $\rho d\mu \in P^\infty(M, \mu)$, the tangent space at $\rho d\mu$ is given by

$$T_{\rho d\mu} P^\infty(M, \mu) = \{s \in C^\infty(M) : \int_M s d\mu = 0\}.$$

By solving the Poisson equation

$$\rho L\phi - \nabla \rho \cdot \nabla \phi = s,$$

there exists a unique function $\phi \in C^\infty(M)$ (up to a constant) such that

$$s = V_\phi := -\nabla_\mu^*(\rho \nabla \phi).$$

Following [18], V_ϕ can be identified as the vector field on $P_2^\infty(M, \mu)$ defined by

$$(V_\phi F)(\rho d\mu) = \left. \frac{\partial}{\partial \varepsilon} \right|_{\varepsilon=0} F(\rho d\mu - \varepsilon \nabla_\mu^*(\rho \nabla \phi) d\mu),$$

where $F \in C^\infty(P^\infty(M))$.

Let $P_2^\infty(M, \mu)$ be the Wasserstein space of probability measures $\rho d\mu \in P^\infty(M)$ with finite second moment

$$\int_M d^2(o, x) \rho d\mu < \infty,$$

where $o \in M$ is a fixed point. Similarly to Otto [24], we can introduce the infinite dimensional Riemannian metric on $T_{\rho d\mu} P_2^\infty(M, \mu)$ as follows

$$\langle \langle s_1, s_2 \rangle \rangle = \int_M \langle \nabla \phi_1, \nabla \phi_2 \rangle \rho d\mu, \quad \forall s_i = V_{\phi_i} \in T_{\rho d\mu} P_2^\infty(M, \mu), \quad i = 1, 2.$$

provided that

$$\int_M |\nabla \phi_i|^2 \rho d\mu < \infty, \quad i = 1, 2.$$

The Wasserstein distance between μ_1 and μ_2 is defined by

$$W_2^2(\mu_1, \mu_2) = \inf_\pi \int_{M \times M} d^2(x, y) d\pi(x, y),$$

where $\pi \in \Pi(\mu_1, \mu_2)$, i.e., π is a probability measure on $M \times M$ such that

$$\int_M \pi(\cdot, dy) = \mu_1, \quad \int_M \pi(dx, \cdot) = \mu_2.$$

The following result is due to Benamou-Brenier [7].

Theorem 2.1 *Let (M, g) be a complete Riemannian manifold. Let $\mu_0 = \rho_0 d\mu$, $\mu_1 = \rho_1 d\mu$ be two probability measures in $P_2(M, \mu)$. Then*

$$W_2^2(\mu_0, \mu_1) = \inf \int_M \int_0^1 |\nabla \phi(x, t)|^2 \rho(x, t) d\mu(x) dt,$$

where $\rho : M \times [0, 1] \rightarrow [0, \infty)$ and $\phi : M \times [0, 1] \rightarrow \mathbb{R}$ satisfy

$$\begin{aligned} \frac{\partial}{\partial t} \rho + \nabla_\mu^*(\rho \nabla \phi) &= 0, \\ \frac{\partial}{\partial t} \phi + \frac{1}{2} |\nabla \phi|^2 &= 0, \\ \rho(\cdot, 0) &= \rho_0, \quad \rho(\cdot, 1) = \rho_1. \end{aligned}$$

The function ϕ in Theorem 2.1 is called the potential function, and $v = \nabla \phi$ is the velocity of the curve $\rho(\cdot, t) dv$ in $P_2^\infty(M, \mu)$. The first equation is called the transport equation, and the second equation is a Hamilton-Jacobi equation. These two equations describe the equation of a geodesic $\rho_s d\mu$ which links $\rho_0 d\mu$ and $\rho_1 d\mu$ in $P_2^\infty(M, \mu)$ equipped with Otto's infinite dimensional Riemannian metric.

In this section, we prove the entropy dissipation formula along the geodesics on the Wasserstein space over a compact Riemannian manifold with weighted volume measure. When the Riemannian manifold is equipped with the nonweighted volume measure, this has been obtained by Lott [18].

Theorem 2.2 *Let (M, g) be a compact Riemannian manifold, $f \in C^2(M)$. Let $\rho : [0, 1] \times M \rightarrow \mathbb{R}$ be a positive solution of the transport equation*

$$\partial_s \rho + \nabla_\mu^*(\rho \nabla \phi) = 0, \tag{23}$$

where $\phi(\cdot, s) : [0, 1] \rightarrow C^\infty(M)$ can be viewed as the velocity vector of the smooth curve $s \rightarrow \rho(s) d\mu$ in $T_{\rho(\cdot) d\mu} P_2^\infty(M, \mu)$. Let

$$\text{Ent}(\rho(s)) := \int_M \rho \log \rho d\mu.$$

Then

$$\frac{d}{ds} \text{Ent}(\rho(s)) = \int_M \langle \nabla \rho, \nabla \phi \rangle d\mu, \tag{24}$$

and

$$\frac{d^2}{ds^2} \text{Ent}(\rho(s)) = - \int_M L\rho(\partial_s \phi + \frac{1}{2} |\nabla \phi|^2) d\mu + \int_M (|\text{Hess} \phi|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi)) \rho d\mu.$$

Proof. By direct calculus and integration by parts formula,

$$\begin{aligned} \frac{d}{ds} \text{Ent}(\rho(s)) &= \int_M \partial_s \rho (1 + \log \rho) d\mu = - \int_M \nabla_\mu^*(\rho \nabla \phi) (1 + \log \rho) d\mu \\ &= \int_M \langle \nabla \rho, \nabla \phi \rangle d\mu = - \int_M L\phi \rho d\mu, \end{aligned} \tag{25}$$

and

$$\begin{aligned}
\frac{d^2}{ds^2} \text{Ent}(\rho(s)) &= \frac{d}{ds} \int_M \langle \nabla \rho, \nabla \phi \rangle d\mu = \int_M \langle \nabla \rho, \nabla \partial_s \phi \rangle d\mu + \int_M \langle \nabla \partial_s \rho, \nabla \phi \rangle d\mu \\
&= \int_M \langle \nabla \rho, \nabla \partial_s \phi \rangle d\mu - \int_M \langle \nabla \nabla_\mu^* (\rho \nabla \phi), \nabla \phi \rangle d\mu \\
&= - \int_M L \rho (\partial_s \phi + \frac{1}{2} |\nabla \phi|^2) d\mu + \int_M \rho L (\frac{1}{2} |\nabla \phi|^2) d\mu - \int_M \langle \nabla \nabla_\mu^* (\rho \nabla \phi), \nabla \phi \rangle d\mu.
\end{aligned}$$

Using again integrating by parts, it holds

$$\int_M \langle \nabla (\nabla_\mu^* (\rho \nabla \phi)), \nabla \phi \rangle d\mu = - \int_M \nabla_\mu^* (\rho \nabla \phi) L \phi d\mu = \int_M \langle \nabla \phi, \nabla L \phi \rangle \rho d\mu.$$

By the weighted Bochner-Weitzenöck formula [2]

$$L |\nabla \phi|^2 = 2 \langle \nabla \phi, \nabla L \phi \rangle + 2 |\text{Hess} \phi|^2 + 2 \text{Ric}(L)(\nabla \phi, \nabla \phi),$$

we can derive that

$$\begin{aligned}
\frac{d^2}{ds^2} \text{Ent}(\rho(s)) &= - \int_M L \rho (\partial_s \phi + \frac{1}{2} |\nabla \phi|^2) d\mu + \int_M \rho L (\frac{1}{2} |\nabla \phi|^2) d\mu - \int_M \rho \langle \nabla \phi, \nabla L \phi \rangle d\mu \\
&= - \int_M L \rho (\partial_s \phi + \frac{1}{2} |\nabla \phi|^2) d\mu + \int_M (|\text{Hess} \phi|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi)) \rho d\mu.
\end{aligned}$$

The proof of Theorem 2.2 is completed. \square

Proposition 2.3 *Let $\rho : [0, 1] \times [0, T] \rightarrow C^\infty(M)$ be a family of positive functions satisfying the transport equation*

$$\partial_s \rho = -\nabla \cdot (\rho \nabla \phi) + \langle \nabla f, \nabla \phi \rangle \rho, \tag{26}$$

where for any fixed $t \in [0, T]$, $\phi(\cdot, t) : [0, 1] \rightarrow C^\infty(M)$ can be viewed as the velocity vector of the smooth curve $s \rightarrow \rho(s, t)$ in $T_{\rho(\cdot, t)d\mu(t)} P^\infty(M)$. Let $c(\cdot, t) = \rho(\cdot, t)d\mu$, and

$$E(c(t)) = \frac{1}{2} \int_0^T \int_0^M |\nabla \phi(s, t)|^2 \rho(s, t) d\mu(x) ds.$$

Then

$$\begin{aligned}
\frac{d}{dt} E(c(t)) &= \int_M \phi \left(\frac{\partial \rho}{\partial t} + L \rho \right) d\mu \Big|_0^T + \int_M |\text{Hess} \phi|^2 \rho d\mu \\
&\quad - \int_0^T \int_M (\partial_s \phi + \frac{1}{2} |\nabla \phi|^2) (\partial_t \rho + L \rho) d\mu ds.
\end{aligned}$$

Proof. Indeed, an elementary calculation shows that

$$\frac{d}{dt} E(c(t)) = \int_0^T \int_M [\langle \nabla \partial_t \phi, \nabla \phi \rangle \rho + \frac{1}{2} |\nabla \phi|^2 \partial_t \rho] d\mu ds.$$

For a fixed function $h \in C^\infty(M)$, it holds

$$\int_M h \frac{\partial \rho}{\partial s} d\mu = \int_M h \nabla_\mu^* \rho d\mu = \int_M \langle \nabla h, \nabla \phi \rangle \rho d\mu.$$

Therefore,

$$\int_M h \frac{\partial^2 \rho}{\partial s \partial t} d\mu = \int_M [\langle \nabla h, \nabla \partial_t \phi \rangle \rho + \langle \nabla h, \nabla \phi \rangle \partial_t \rho] d\mu.$$

In particular, taking $h = \phi$, we obtain

$$\int_M \phi \frac{\partial^2 \rho}{\partial s \partial t} d\mu = \int_M [\langle \nabla \phi, \nabla \partial_t \phi \rangle \rho + |\nabla \phi|^2 \partial_t \rho] d\mu.$$

Thus

$$\begin{aligned} \frac{d}{dt} E(c(t)) &= \int_0^T \int_M [\langle \nabla \partial_t \phi, \nabla \phi \rangle \rho + \frac{1}{2} |\nabla \phi|^2 \partial_t \rho] d\mu ds \\ &= \int_0^T \int_M [\phi \frac{\partial^2 \rho}{\partial s \partial t} - \frac{1}{2} |\nabla \phi|^2 \partial_t \rho] d\mu ds \\ &= \int_M \phi \partial_t \rho d\mu \Big|_0^T - \int_0^T \int_M (\partial_s \phi + \frac{1}{2} |\nabla \phi|^2) \partial_t \rho d\mu ds. \end{aligned}$$

Applying Theorem 2.2 to the transport equation (26) on $(M, g(t), f(t))$, we have

$$\frac{d^2}{ds^2} \text{Ent}(\rho(s, t)) = - \int_M L\rho (\partial_s \phi + \frac{1}{2} |\nabla \phi|^2) d\mu + \int_M (|\text{Hess} \phi|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi)) \rho d\mu.$$

Integrating from $s = 0$ to $s = T$, and using (24), we obtain

$$\int_M \langle \nabla \rho, \nabla \phi \rangle d\mu \Big|_0^T = - \int_0^T \int_M L\rho (\partial_s \phi + \frac{1}{2} |\nabla \phi|^2) d\mu ds + \int_0^T \int_M (|\text{Hess} \phi|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi)) \rho d\mu ds.$$

Combining the last equation with the previous one for $\frac{d}{dt} E(c(t))$, we have

$$\begin{aligned} \frac{d}{dt} E(c(t)) &= \int_M \phi \left(\frac{\partial \rho}{\partial t} + L\rho \right) d\mu \Big|_0^T + \int_M |\text{Hess} \phi|^2 \rho d\mu \\ &\quad - \int_0^T \int_M \left(\partial_s \phi + \frac{1}{2} |\nabla \phi|^2 \right) (\partial_t \rho + L\rho) d\mu ds. \end{aligned}$$

The proof of Proposition 2.3 is completed. \square

As a consequence of Theorem 2.2, we have the following

Theorem 2.4 *Suppose that (ρ, ϕ) satisfies the geodesic equation on $P_2^\infty(M, \mu)$*

$$\begin{aligned} \frac{\partial \phi}{\partial s} + \frac{1}{2} |\nabla \phi|^2 &= 0, \\ \frac{\partial \rho}{\partial s} + \nabla_\mu^* (\rho \nabla \phi) &= 0. \end{aligned}$$

Then

$$\frac{d^2}{ds^2} \text{Ent}(\rho(s)) = \int_M (|\text{Hess}\phi|^2 + \text{Ric}(L)(\nabla\phi, \nabla\phi)) \rho d\mu.$$

In view of Theorem 2.4, the Hessian of the Boltzmann entropy functional Ent on $P_2^\infty(M, \mu)$ equipped with Otto's infinite dimensional Riemannian metric is given by

$$\text{Hess}_{P_2^\infty(M, \mu)} \text{Ent}(V_\phi, V_\phi) = \int_M (|\text{Hess}\phi|^2 + \text{Ric}(L)(\nabla\phi, \nabla\phi)) \rho d\mu. \quad (27)$$

As a corollary of Theorem 2.4, we have the following result due to Lott-Villani [19, 18], Sturm-von Renesse [34] and Sturm [32, 33].

Corollary 2.5 *If $\text{Ric}(L) \geq 0$, then $\frac{d^2}{ds^2} \text{Ent}(\rho(s)) \geq 0$, i.e., Ent is convex along geodesic in $P_2^\infty(M, \mu)$.*

3 W -entropy formula for heat equation of Witten Laplacian

In this section, for the convenience of the reader, we review the W -entropy formula for the heat equation of the Witten Laplacian on compact or complete Riemannian manifolds with weighted volume measure. The main result, i.e., Theorem 1.3, has been proved in our previous papers [12, 13, 14, 16]. Here we give an alternative proof of Theorem 1.3.

3.1 Case of compact Riemannian manifolds

In this section we give an alternative proof of Theorem 1.1 as follows: Note that $\dot{\rho} = \nabla \text{Ent}(\rho) = -L\rho$. This yields

$$\frac{d}{dt} \text{Ent}(\rho(t)) = \nabla \text{Ent}(\rho(t)) \cdot \dot{\rho}(t) = |\nabla \text{Ent}(\rho(t))|^2.$$

By Otto's calculus on the Wasserstein space over M , we have

$$\frac{d}{dt} \text{Ent}(\rho(t)) = \int_M \frac{|\nabla \rho|^2}{\rho} d\mu.$$

Moreover

$$\begin{aligned} \frac{d^2}{dt^2} \text{Ent}(\rho(t)) &= \nabla^2 \text{Ent}(\rho(t))(\dot{\rho}(t), \dot{\rho}(t)) + \nabla \text{Ent}(\rho(t)) \cdot \ddot{\rho}(t) \\ &= \nabla^2 \text{Ent}(\rho(t))(\dot{\rho}(t), \dot{\rho}(t)) + \nabla \text{Ent}(\rho) \cdot \nabla^2 \text{Ent}(\rho) \cdot \dot{\rho}(t) \\ &= 2\nabla^2 \text{Ent}(\rho(t))(\dot{\rho}(t), \dot{\rho}(t)). \end{aligned}$$

Theorem 2.4 yields

$$\frac{d^2}{dt^2} \text{Ent}(\rho(t))(\dot{\rho}, \dot{\rho}) = 2 \int_M [|\text{Hess} \log \rho|^2 + \text{Ric}(L)(\nabla \log \rho, \nabla \log \rho)] \rho d\mu.$$

Thus

$$\begin{aligned}
& \frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \frac{2}{t} \frac{d}{dt} \text{Ent}(\rho(t)) + \frac{m}{2t^2} \\
&= 2 \int_M (|\text{Hess} \log \rho|^2 + \text{Ric}(L)(\nabla \log \rho, \nabla \log \rho)) \rho d\mu - \frac{2}{t} \int_M L \log \rho \rho d\mu + \frac{m}{2t^2} \\
&= 2 \int_M \left(\left| \text{Hess} \log \rho - \frac{g}{2t} \right|^2 + \frac{1}{t} \Delta \log \rho + \text{Ric}(L)(\nabla \log \rho, \nabla \log \rho) \right) \rho d\mu - \frac{2}{t} \int_M L \log \rho \rho d\mu + \frac{m-n}{2t^2} \\
&= 2 \int_M \left| \text{Hess} \log \rho - \frac{g}{2t} \right|^2 \rho d\mu + 2 \int_M \text{Ric}(L)(\nabla \log \rho, \nabla \log \rho) \rho d\mu + \frac{2}{t} \int_M \nabla \log \rho \cdot \nabla f \rho d\mu + \frac{m-n}{2t^2}.
\end{aligned}$$

Note that

$$\begin{aligned}
& 2\text{Ric}(L)(\nabla \log \rho, \nabla \log \rho) + \frac{2}{t} \nabla \log \rho \cdot \nabla f + \frac{m-n}{2t^2} \\
&= 2\text{Ric}_{m,n}(\nabla \log \rho, \nabla \log \rho) + 2 \frac{|\nabla \log \rho \cdot \nabla f|^2}{m-n} + \frac{2}{t} \nabla \log \rho \cdot \nabla f + \frac{m-n}{2t^2} \\
&= 2\text{Ric}_{m,n}(\nabla \log \rho, \nabla \log \rho) + \frac{2}{m-n} \left| \nabla \log \rho \cdot \nabla f + \frac{m-n}{2t} \right|^2.
\end{aligned}$$

Thus

$$\begin{aligned}
& \frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \frac{2}{t} \frac{d}{dt} \text{Ent}(\rho(t)) + \frac{m}{2t^2} \\
&= 2 \int_M \left[\left| \text{Hess} \log \rho - \frac{g}{2t} \right|^2 + \text{Ric}_{m,n}(L)(\nabla \log \rho, \nabla \log \rho) \right] \rho d\mu \\
& \quad + \frac{2}{m-n} \int_M \left| \nabla \log \rho \cdot \nabla f + \frac{m-n}{2t} \right|^2 \rho d\mu.
\end{aligned}$$

This completes the proof of Theorem 1.3. \square

In the case of compact Riemannian manifolds with the $CD(K, N)$ condition, we have the following

Theorem 3.1 *Let M be a compact Riemannian manifold. Suppose that the $CD(K, m)$ condition holds. The, for the backward gradient flow $\dot{\rho}(t) = \nabla \text{Ent}(\rho(t)) = -L\rho$ on $P_2(M, \mu)$, we have*

$$\begin{aligned}
& \frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \frac{2}{t} \frac{d}{dt} \text{Ent}(\rho(t)) + \frac{m}{2} \left(K + \frac{1}{t} \right)^2 \\
&= 2 \int_M \left[\left| \text{Hess} \log \rho - \frac{1}{2} \left(K + \frac{1}{t} \right) g \right|^2 + (\text{Ric}_{m,n}(L) - K)(\nabla \log \rho, \nabla \log \rho) \right] \rho d\mu \\
& \quad + \frac{2}{m-n} \int_M \left(\nabla f \cdot \nabla \log \rho + \frac{m-n}{2} \left(K + \frac{1}{t} \right) \right)^2 \rho d\mu. \tag{28}
\end{aligned}$$

Proof. Indeed, by result in Section 2, we have

$$\begin{aligned}
& \frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \frac{2}{t} \frac{d}{dt} \text{Ent}(\rho(t)) + \frac{m}{2} \left(K + \frac{1}{t} \right)^2 \\
&= 2 \text{HessEnt}(\rho(t)) + \frac{2}{t} \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle + \frac{m}{2} \left(K + \frac{1}{t} \right)^2 \\
&= 2 \int_M [|\text{Hess} \log \rho|^2 + \text{Ric}(L)(\nabla \log \rho, \nabla \log \rho)] \rho d\mu + \frac{2}{t} \int_M |\nabla \log \rho|^2 \rho d\mu + \frac{m}{2} \left(K + \frac{1}{t} \right)^2 \\
&= 2 \int_M [|\text{Hess} \log \rho - \alpha(t)g|^2 + \text{Ric}(L)(\nabla \log \rho, \nabla \log \rho)] \rho d\mu \\
&\quad + 4\alpha(t) \int_M \Delta \log \rho \rho d\mu - 2n\alpha^2(t) + \frac{2}{t} \int_M |\nabla \log \rho|^2 \rho d\mu + \frac{m}{2} \left(K + \frac{1}{t} \right)^2 \\
&= 2 \int_M [|\text{Hess} \log \rho - \alpha(t)g|^2 + (\text{Ric}_{m,n}(L) - K)(\nabla \log \rho, \nabla \log \rho)] \rho d\mu \\
&\quad + \frac{2}{N-n} \int_M |\nabla f \cdot \nabla \log \rho|^2 \rho d\mu + 4\alpha(t) \int_M \Delta \log \rho \rho d\mu \\
&\quad - 2n\alpha^2(t) + 2 \left(K + \frac{1}{t} \right) \int_M |\nabla \log \rho|^2 \rho d\mu + \frac{m}{2} \left(K + \frac{1}{t} \right)^2.
\end{aligned}$$

Integration by parts yields

$$\int_M |\nabla \log \rho|^2 \rho d\mu = - \int_M L \log \rho \rho d\mu = - \int_M (\Delta \log \rho - \nabla f \cdot \nabla \log \rho) \rho d\mu.$$

Taking $\alpha(t) = \frac{1}{2} \left(K + \frac{1}{t} \right)$, we have

$$\begin{aligned}
& \frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \frac{2}{t} \frac{d}{dt} \text{Ent}(\rho(t)) + \frac{m}{2} \left(K + \frac{1}{t} \right)^2 \\
&= 2 \int_M \left[|\text{Hess} \log \rho - \frac{1}{2} \left(K + \frac{1}{t} \right) g|^2 + (\text{Ric}_{m,n}(L) - K)(\nabla \log \rho, \nabla \log \rho) \right] \rho d\mu \\
&\quad + \frac{2}{m-n} \int_M |\nabla f \cdot \nabla \log \rho|^2 \rho d\mu + 2 \left(K + \frac{1}{t} \right) \int_M \nabla f \cdot \nabla \log \rho \rho d\mu + \frac{m-n}{2} \left(K + \frac{1}{t} \right)^2 \\
&= 2 \int_M \left[|\text{Hess} \log \rho - \frac{1}{2} \left(K + \frac{1}{t} \right) g|^2 + (\text{Ric}_{m,n}(L) - K)(\nabla \log \rho, \nabla \log \rho) \right] \rho d\mu \\
&\quad + \frac{2}{m-n} \int_M \left(\nabla f \cdot \nabla \log \rho + \frac{m-n}{2} \left(K + \frac{1}{t} \right) \right)^2 \rho d\mu.
\end{aligned}$$

This finishes the proof. \square

Remark 3.2 In [14, 16], the authors proved the W -entropy formula for the forward heat equation $\partial_t u = Lu$ for the Witten Laplacian on complete Riemannian manifolds with $CD(K, m)$ condition. Theorem 3.1 is an analogue of our previous result for the backward heat equation $\partial_t \rho = -L\rho$, which is the backward gradient flow of the Boltzmann entropy on $P_2(M, \mu)$.

Remark 3.3 Taking trace in the right hand side of (28) and applying the elementary inequality $(a+b)^2 \geq \frac{a^2}{1+\varepsilon} - \frac{b^2}{\varepsilon}$ to $a = L \log \rho - \frac{m-n}{2} \left(K + \frac{1}{t}\right)$, $b = \nabla f \cdot \nabla \log \rho + \frac{m-n}{2} \left(K + \frac{1}{t}\right)$ and $\varepsilon = \frac{m-n}{n}$, under the condition $CD(K, m)$, we have

$$\begin{aligned} & \frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \frac{2}{t} \frac{d}{dt} \text{Ent}(\rho(t)) + \frac{m}{2} \left(K + \frac{1}{t}\right)^2 \\ & \geq \frac{2}{n} \int_M \left[\Delta \log \rho - \frac{n}{2} \left(K + \frac{1}{t}\right) \right]^2 \rho d\mu + \frac{2}{m-n} \int_M \left(\nabla f \cdot \nabla \log \rho + \frac{m-n}{2} \left(K + \frac{1}{t}\right) \right)^2 \rho d\mu \\ & \geq \frac{2}{m} \int_M \left[L \log \rho - \frac{m}{2} \left(K + \frac{1}{t}\right) \right]^2 \rho d\mu. \end{aligned}$$

By the Cauchy-Schwartz inequality, we have

$$\begin{aligned} & \frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \frac{2}{t} \frac{d}{dt} \text{Ent}(\rho(t)) + \frac{m}{2} \left(K + \frac{1}{t}\right)^2 \\ & \geq \frac{2}{m} \left(\int_M \left[L \log \rho - \frac{m}{2} \left(K + \frac{1}{t}\right) \right] \rho d\mu \right)^2. \end{aligned}$$

Integration by parts yields

$$\frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \frac{2}{t} \frac{d}{dt} \text{Ent}(\rho(t)) + \frac{m}{2} \left(K + \frac{1}{t}\right)^2 \geq \frac{2}{m} \left[\int_M |\nabla \log \rho|^2 \rho d\mu + \frac{m}{2} \left(K + \frac{1}{t}\right)^2 \right]. \quad (29)$$

In Section 8.1, we will derive the inequality (29) under Erbar-Kuwada-Sturm's entropic curvature-dimension condition $CD_{\text{Ent}}(K, m)$, which is weaker than Bakry-Emery's $CD(K, m)$ -condition. See Theorem 8.1.(ii).

3.2 Case of complete Riemannian manifolds

In our previous papers [12, 13, 14, 16], we extend the W -entropy formula for the heat equation of the Witten Laplacian to complete Riemannian manifolds with bounded geometry condition. More precisely, we have the following

Theorem 3.4 ([12, 13, 14, 16]) *Let (M, g) be a complete Riemannian manifold with bounded geometry condition³, and $f \in C^4(M)$ with $\nabla f \in C_b^3(M)$. Let u be the heat kernel to the heat equation $\partial_t u = Lu$. Let.*

$$H_m(u, t) = \int_M u \log u d\mu + \frac{m}{2} (1 + \log(4\pi t)).$$

Define

$$W_m(u, t) = \frac{d}{dt} (t H_m(u)).$$

³Here we say that (M, g) satisfies the bounded geometry condition if the Riemannian curvature tensor Riem and its covariant derivatives $\nabla^k \text{Riem}$ are uniformly bounded on M , $k = 1, 2, 3$.

Then

$$\begin{aligned} \frac{d}{dt} W_m(u, t) &= 2 \int_M t \left(\left| \nabla^2 \log u + \frac{g}{2t} \right|^2 + Ric_{m,n}(L)(\nabla \log u, \nabla \log u) \right) u d\mu \\ &\quad + \frac{2}{m-n} \int_M t \left(\nabla \phi \cdot \nabla \log u - \frac{m-n}{2t} \right)^2 u d\mu. \end{aligned} \quad (30)$$

3.3 Monotonicity and rigidity theorem for the W -entropy for heat equation

As a consequence of Theorem 3.4, we have the following monotonicity and rigidity theorem for W -entropy for the heat equation of the Witten Laplacian on complete Riemannian manifolds with bounded geometry condition.

Theorem 3.5 ([13, 14, 16]) *Let (M, g) be a complete Riemannian manifold with bounded geometry condition and $f \in C^4(M)$ with $\nabla f \in C_b^3(M)$. Suppose that $Ric_{m,n}(L) \geq 0$. Then the W -entropy is non-decreasing in time $t \in \mathbb{R}^+$, i.e.,*

$$\frac{dW_m(u, t)}{dt} \geq 0, \quad \forall t \geq 0.$$

Moreover

$$\frac{dW_m(u, t)}{dt} = 0, \quad \text{at some } t = t_0 > 0,$$

if and only if $M = \mathbb{R}^n$, $m = n$, f is a constant, and $u(x, t) = \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}}$ for all $x \in M, t > 0$.

Proof. See [13]. □

4 W -entropy formula for geodesic flow on Wasserstein space

In this section, we introduce the W -entropy functional and prove the W -entropy formula for the geodesic flow on the Wasserstein space over compact Riemannian manifolds with weighted volume measure, i.e., Theorem 1.1. We will also compare the W -entropy formula in Theorem 1.1 with the W -entropy formula for the heat equation of the Witten Laplacian on compact Riemannian manifolds (i.e., Theorem 1.3), and then introduce the W -entropy for the optimal transport problem on compact or complete Riemannian manifolds with weighted volume measure.

4.1 Proof of Theorem 1.1

By (25) and (27), we have

$$\begin{aligned} \frac{d}{dt} \text{Ent}(\rho(t)) &= - \int_M L\phi \rho d\mu, \\ \frac{d^2}{dt^2} \text{Ent}(\rho(t)) &= \int_M (|\text{Hess}\phi|^2 + Ric(L)(\nabla\phi, \nabla\phi)) \rho d\mu. \end{aligned}$$

Thus

$$\begin{aligned}
\frac{d}{dt}W_m(\rho, t) &= t \frac{d^2}{dt^2} \text{Ent}(\rho(t)) + 2 \frac{d}{dt} \text{Ent}(\rho(t)) + \frac{m}{t} \\
&= t \int_M (|\text{Hess}\phi|^2 + \text{Ric}(L)(\nabla\phi, \nabla\phi)) \rho d\mu - 2 \int_M L\phi \rho d\mu + \frac{m}{t} \\
&= t \int_M \left(\left| \text{Hess}\phi - \frac{g}{t} \right|^2 + \frac{2}{t} \Delta\phi + \text{Ric}(L)(\nabla\phi, \nabla\phi) \right) \rho d\mu - t \int_M L\phi \rho d\mu + \frac{m-n}{t} \\
&= t \int_M \left[\left| \text{Hess}\phi - \frac{g}{t} \right|^2 + \text{Ric}(L)(\nabla\phi, \nabla\phi) \right] \rho d\mu + 2 \int_M \nabla\phi \cdot \nabla f \rho d\mu + \frac{m-n}{t}.
\end{aligned}$$

Note that

$$\begin{aligned}
&\text{Ric}(L)(\nabla\phi, \nabla\phi) + \frac{2}{t} \nabla\phi \cdot \nabla f + \frac{m-n}{t^2} \\
&= \text{Ric}_{m,n}(\nabla\phi, \nabla\phi) + \frac{|\nabla\phi \cdot \nabla f|^2}{m-n} + \frac{2}{t} \nabla\phi \cdot \nabla f + \frac{m-n}{t^2} \\
&= \text{Ric}_{m,n}(\nabla\phi, \nabla\phi) + \frac{1}{m-n} \left| \nabla\phi \cdot \nabla f + \frac{m-n}{t} \right|^2.
\end{aligned}$$

Thus

$$\frac{d}{dt}W_m(\rho, t) = t \int_M \left[\left| \text{Hess}\phi - \frac{g}{t} \right|^2 + \text{Ric}_{m,n}(L)(\nabla\phi, \nabla\phi) \right] \rho d\mu + \frac{t}{m-n} \int_M \left| \nabla\phi \cdot \nabla f + \frac{m-n}{t} \right|^2 \rho d\mu.$$

This proves the W -entropy formula (4) in Theorem 1.1. \square

As a corollary of Theorem 1.1, we can recapture the following result due to Lott-Villani [19]. See also Lott [18].

Corollary 4.1 (*i.e., Corollary 1.2*) *If $\text{Ric}_{m,n}(L) \geq 0$, then $t\text{Ent} + mt \log t$ is convex in t along the geodesic flow in $T^*P_2^\infty(M, \mu)$.*

4.2 Case of compact Riemannian manifolds

Let $\rho_n(t, x) = \frac{e^{-\frac{\|x\|^2}{4t^2}}}{(4\pi t^2)^{n/2}}$ be the special solution of the transport equation (1) with the velocity $\phi_m(t, x) = \frac{\|x\|^2}{2t}$ on the Euclidean space \mathbb{R}^n . By calculus, the Boltzmann-Shannon entropy of $\rho_n(t, x)dx$ with respect to the Lebesgue measure on \mathbb{R}^n is given by

$$\text{Ent}(\rho_n(t)) = -\frac{n}{2}(1 + \log(4\pi t^2)).$$

Now, for the optimal transport problem on compact Riemannian manifolds (M, g) with weighted measure $d\mu = e^{-f} dv$, the geodesic flow on $P_2^\infty(M, \mu)$ equipped with Otto's infinite dimensional Riemannian metric is given by the transport equation together with the Hamilton-Jacobi equation

$$\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 = 0, \tag{31}$$

$$\partial_t \rho + \nabla_\mu^*(\rho \nabla \phi) = 0. \tag{32}$$

Define the Boltzmann-Shannon entropy

$$\text{Ent}(\rho(t)) = \int_M \rho \log \rho d\mu, \quad (33)$$

By Theorem 2.4, we have

$$\begin{aligned} \frac{d}{dt} \text{Ent}(\rho(t)) &= - \int_M \langle \nabla \rho, \nabla \phi \rangle d\mu = \int_M L\phi \rho d\mu, \\ \frac{d^2}{dt^2} \text{Ent}(\rho(t)) &= \int_M [|\nabla^2 \phi|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi)] \rho d\mu. \end{aligned}$$

Following Perelman [29], we introduce the W -entropy associated with the optimal transport problem on (M, g, μ) or the geodesic flow on $T^*P_2(M, \mu)$ as follows

$$W_m(\rho, t) := \frac{d}{dt} (tH_m(\rho, t)), \quad (34)$$

where

$$H_m(\rho, t) = \text{Ent}(\rho(t)) - \text{Ent}(\rho_m(t)) \quad (35)$$

is the difference between the Boltzmann-Shannon entropy of the probability measure $\rho d\mu$ on (M, g) and the Boltzmann-Shannon entropy of the probability measure $\rho_m(t, x)dx$ on \mathbb{R}^m . Substituting (34) into (35) and (34) we have

$$W_m(\rho, t) = \int_M [tL\phi - \log \rho - \text{Ent}(\rho_m(t))] \rho d\mu. \quad (36)$$

Moreover, we can reformulate Theorem 1.1 as follows

$$\begin{aligned} \frac{d}{dt} W_m(\rho, t) &= t \int_M \left[\left| \text{Hess} \phi - \frac{g}{t} \right|^2 + \text{Ric}_{m,n}(L)(\nabla \phi, \nabla \phi) \right] \rho d\mu \\ &\quad + \frac{t}{m-n} \int_M \left| \nabla \phi \cdot \nabla f + \frac{m-n}{t} \right|^2 \rho d\mu. \end{aligned}$$

In view of this reformulation, we have the following

Theorem 4.2 *Let M be a compact Riemannian manifold. Let ϕ and ρ be a smooth solution to the Hamilton-Jacobi equation (31) and the transport equation (32). Let*

$$W_m(\rho, t) = \frac{d}{dt} (t[\text{Ent}(\rho(t)) - \text{Ent}(\rho_m(t))])$$

be the W -entropy associated to the optimal transport on (M, dv) . Then

$$\begin{aligned} \frac{\partial}{\partial t} W_m(\rho, t) &= t \int_M \left[\left| \text{Hess} \phi - \frac{g}{t} \right|^2 + \text{Ric}_{m,n}(L)(\nabla \phi, \nabla \phi) \right] \rho d\mu \\ &\quad + \frac{t}{m-n} \int_M \left| \nabla \phi \cdot \nabla f - \frac{m-n}{t} \right|^2 \rho d\mu. \end{aligned}$$

*In particular, if $\text{Ric}_{m,n}(L) \geq 0$, then the Helmholtz free energy $S_m = t(\text{Ent}(\rho(t)) - \text{Ent}(\rho_m(t)))$ associated with (31) and (32) is convex in time t along the geodesic flow on $T^*P_2^\infty(M, \mu)$.*

Corollary 4.3 (i.e., Corollary 1.2, [19, 18]) *Let M be a compact Riemannian manifold. Suppose that $\text{Ric}_{m,n}(L) \geq 0$. Then $t\text{Ent}(\rho(t)) + mt \log t$ is convex in time t along the geodesic flow on $T^*P_2^\infty(M, \mu)$.*

Proof. By Theorem 1.1, if $\text{Ric}_{m,n}(L) \geq 0$, $t(\text{Ent}(\rho(t)) - \text{Ent}(\rho_m(t))) = t\text{Ent}(\rho(t)) + \frac{mt}{2}[\log(4\pi t^2) + 1]$ is convex in t along the geodesic flow on $T^*P_2^\infty(M, \mu)$. Note that

$$\frac{d^2}{dt^2}(t(\text{Ent}(\rho(t)) - \text{Ent}(\rho_m(t)))) = \frac{d^2}{dt^2}(t\text{Ent}(\rho(t)) + mt \log t).$$

Hence $t\text{Ent} + mt \log t$ is convex in t along the geodesic flow on $T^*P_2^\infty(M, \mu)$. For the general case of non smooth geodesic flow on $P_2(M, \mu)$, see Lott [18]. \square

In particular, taking $f = 0$, $m = n$ and $L = \Delta$, we have the following result which is also new in the literature.

Theorem 4.4 *Let M be a compact Riemannian manifold. Let ϕ and ρ be a smooth solution to the Hamilton-Jacobi equation and the transport equation*

$$\partial_t \phi + \frac{1}{2}|\nabla \phi|^2 = 0, \quad (37)$$

$$\partial_t \rho + \text{div}(\rho \nabla \phi) = 0. \quad (38)$$

Let

$$W_n(\rho, t) = \frac{d}{dt}(t[\text{Ent}(\rho(t)) - \text{Ent}(\rho_n(t))]). \quad (39)$$

Then

$$\frac{\partial}{\partial t} W_n(\rho, t) = t \int_M \left[\left| \text{Hess} \phi - \frac{g}{t} \right|^2 + \text{Ric}(\nabla \phi, \nabla \phi) \right] \rho d\mu. \quad (40)$$

In particular, if $\text{Ric} \geq 0$, then the W -entropy W_n associated with (37) and (38) is increasing in time t , and $t\text{Ent} + nt \log t$ is convex in t along the geodesic flow on $T^*P_2(M, \nu)$.

4.3 Case of complete Riemannian manifolds

To extend Theorem 1.1 to complete Riemannian manifolds with bounded geometry condition, we need the following

Theorem 4.5 *Let M be a complete Riemannian manifold, and $f \in C^2(M)$. Suppose that $\text{Ric}(L) = \text{Ric} + \nabla^2 f$ is uniformly bounded on M , i.e., there exists a constant $C > 0$ such that $|\text{Ric}(L)| \leq C$. Let ρ and ϕ be smooth solutions to the transport equation (1) and the Hamilton-Jacobi equation (2), and satisfying the following growth conditions*

$$\int_M [|\nabla \log \rho|^2 + |\nabla \phi|^2 + |\nabla^2 \phi|^2 + |L\phi|^2 + |\nabla L\phi|^2] \rho d\mu < \infty, \quad (41)$$

and there exist a point $o \in M$, and some functions $C_i \in C([0, T], \mathbb{R}^+)$ and $\alpha_i \in C([0, T], \mathbb{R}^+)$ such that

$$C_1(t)e^{-\alpha_1(t)d^2(x,o)} \leq \rho(x, t) \leq C_2(t)e^{\alpha_2(t)d^2(x,o)}, \quad \forall x \in M, t \in [0, T], \quad (42)$$

and

$$\int_M d^4(x, o) \rho(x, t) d\mu < \infty, \quad \forall t \in [0, T].$$

Then the entropy dissipation formulas hold

$$\partial_t \int_M \rho \log \rho d\mu = \int_M \nabla \phi \cdot \nabla \rho d\mu = - \int_M L\phi \rho d\mu, \quad (43)$$

$$\partial_t^2 \int_M \rho \log \rho d\mu = \int_M [|\nabla \phi|^2 + Ric(L)(\nabla \phi, \nabla \phi)] \rho d\mu. \quad (44)$$

Proof. Let η_k be an increasing sequence of functions in $C_0^\infty(M)$ such that $0 \leq \eta_k \leq 1$, $\eta_k = 1$ on $B(o, k)$, $\eta_k = 0$ on $M \setminus B(o, 2k)$, and $\|\nabla \eta_k\| \leq \frac{1}{k}$. By standard argument and integration by parts, we have

$$\begin{aligned} \partial_t \int_M \rho \log \rho \eta_k d\mu &= \int_M \partial_t(\rho \log \rho) \eta_k d\mu = \int_M \partial_t \rho (\log \rho + 1) \eta_k d\mu \\ &= \int_M \nabla_\mu^*(\rho \nabla \phi) (\log \rho + 1) \eta_k d\mu \\ &= \int_M \rho \nabla \phi \cdot \nabla \log \rho \eta_k d\mu + \int_M \rho \nabla \phi \cdot (\log \rho + 1) \nabla \eta_k d\mu \\ &= I_1 + I_2. \end{aligned} \quad (45)$$

Under the conditions (41), (42), (43), the Lebesgue dominated convergence theorem yields

$$\begin{aligned} I_1 &= \int_M \rho \nabla \phi \cdot \nabla \log \rho \eta_k d\mu \rightarrow \int_M \nabla \phi \cdot \nabla \rho d\mu, \\ I_1 &= - \int_M \nabla_\mu^*(\eta_k \nabla \phi) \rho d\mu = - \int_M \eta_k L\phi \rho d\mu - \int_M \nabla \eta_k \cdot \nabla \phi \rho d\mu \rightarrow - \int_M L\phi \rho, \\ I_2 &= \int_M \rho \nabla \phi \cdot (\log \rho + 1) \nabla \eta_k d\mu \rightarrow 0. \end{aligned}$$

Taking $k \rightarrow \infty$ in (45), we can derive (43).

Furthermore, standard argument yields

$$\begin{aligned} \partial_t \int_M L\phi \rho \eta_k d\mu &= \int_M \partial_t(L\phi \rho) \eta_k d\mu \\ &= \int_M L\partial_t \phi \rho \eta_k d\mu + \int_M L\phi \partial_t \rho \eta_k d\mu \\ &= \int_M L \left(\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 \right) \rho \eta_k d\mu - \frac{1}{2} \int_M L |\nabla \phi|^2 \rho \eta_k d\mu + \int_M L\phi \partial_t \rho \eta_k d\mu \\ &= -\frac{1}{2} \int_M L |\nabla \phi|^2 \rho \eta_k d\mu - \int_M L\phi \nabla_\mu^*(\rho \nabla \phi) \eta_k d\mu \\ &= I_3 + I_4. \end{aligned} \quad (46)$$

By the Bochner formula we have

$$\int_M |L|\nabla \phi|^2| \rho d\mu = 2 \int_M |\nabla \phi \cdot \nabla L\phi + |\nabla^2 \phi|^2 + Ric(L)(\nabla \phi, \nabla \phi)| \rho d\mu.$$

Suppose that $|Ric(L)| \leq C$, and $\int_M [|\nabla\phi|^2 + |\nabla L\phi|^2 + |\nabla^2\phi|^2 + |\nabla\phi|^2]\rho d\mu < \infty$. Then

$$\begin{aligned} \int_M |L|\nabla\phi|^2|\rho d\mu &= 2 \int_M |\nabla\phi \cdot \nabla L\phi + |\nabla^2\phi|^2 + Ric(L)(\nabla\phi, \nabla\phi)| \rho d\mu \\ &\leq 2 \int_M [|\nabla\phi||\nabla L\phi| + |\nabla^2\phi|^2 + C|\nabla\phi|^2] \rho d\mu < \infty. \end{aligned}$$

The Lebesgue dominated convergence theorem yields

$$I_3 \rightarrow -\frac{1}{2} \int_M L|\nabla\phi|^2 \rho d\mu.$$

On the other hand, if $\int_M [|\nabla\phi|^2 + |\nabla L\phi|^2] \rho d\mu < \infty$, then

$$\begin{aligned} I_4 &= - \int_M L\phi \nabla_\mu^* (\rho \nabla\phi) \eta_k d\mu = \int_M \nabla(\eta_k L\phi) \cdot \rho \nabla\phi d\mu \\ &= \int_M [L\phi \nabla \eta_k \cdot \nabla\phi + \eta_k \nabla L\phi \cdot \nabla\phi] \rho d\mu \rightarrow \int_M \nabla L\phi \cdot \nabla\phi \rho d\mu. \end{aligned}$$

Taking $k \rightarrow \infty$ in (46), we can derive (44). This finishes the proof of Theorem 4.5. \square

Based on Theorem 4.5, Theorem 1.1 can be extended to complete Riemannian manifolds as follows.

Theorem 4.6 *Let M be a complete Riemannian manifold with bounded geometry condition. Under the same conditions as in Theorem 4.5, the W -entropy formula in Theorem 1.1 remains true.*

Proof. The proof is similar to the one of Theorem 1.1. \square

4.4 Monotonicity and rigidity theorem for the W -entropy for geodesic flow

Similarly to the monotonicity and rigidity theorem (i.e., Theorem 3.5) of the W -entropy for the heat equation associated with the Witten Laplacian (which is the gradient flow of the Boltzmann entropy on the Wasserstein space) over complete Riemannian manifolds with bounded geometry condition, we have the following monotonicity and rigidity theorem of the W -entropy for the geodesic flow on the Wasserstein space over complete Riemannian manifolds with bounded geometry condition.

Theorem 4.7 *Let M be a complete Riemannian manifold with bounded geometry condition and with $Ric_{m,n}(L) \geq 0$. Suppose that (ρ, ϕ) is a smooth solution to the transport equation (1) and the Hamilton-Jacobi equation (2) satisfying the growth condition as required in Theorem 4.5. Then $W_m(\rho, t)$ is increasing in t , and $t\text{Ent}(\rho(t)) + mt \log t$ is convex in t . Moreover, $\frac{d}{dt} W_m(\rho, t) = 0$ holds at some $t = t_0$ if and only if M is isometric to \mathbb{R}^n , $n = m$, and $(\rho, \phi) = (\rho_n, \phi_m)$. That is to say, the Euclidean space \mathbb{R}^n equipped with the Gaussian distribution $N(0, t^2 \text{Id})$ is the rigidity model for the W -entropy for the geodesic flow on the Wasserstein space over complete Riemannian manifolds with $CD(0, m)$ -condition, where Id is the unit matrix on \mathbb{R}^m .*

Proof. The proof is as the same as the one of Theorem 2.5 in [13]. For the convenience of the reader, we give the detail here. Indeed, by the explicit expression of $\frac{d}{dt}W_m(\rho, t)$ in Theorem 1.1, we see that $\frac{d}{dt}W_m(\rho, t) = 0$ holds at some $t = t_0$ if and only if $\nabla^2\phi(\cdot, t_0) = \frac{g}{t_0}$. Thus, ϕ is a strict convex function on M , which implies that M is diffeomorphic to \mathbb{R}^n . Integrating along the shortest geodesics between x_0 and x on M shows that

$$2t_0(\phi(x, t_0) - \phi(x_0, t_0)) = r^2(x_0, x), \quad \forall x \in M,$$

where x_0 is the minimum point of $\phi(\cdot, \tau)$. This yields

$$\Delta r^2(x_0, x) = 2n, \quad \forall x \in M,$$

which implies that (M, g) is isometric to the Euclidean space $(\mathbb{R}^n, (\delta_{ij}))$. By the generalized Cheeger-Gromoll splitting theorem (see Theorem 1.3, p. 565, [6]), we can derive that f must be a constant and $m = n$.

Thus $\phi(\cdot, t_0) \in C^\infty(\mathbb{R}^n)$ satisfies $\nabla^2\phi(x, t_0) = \frac{\delta_{ij}}{t_0}$. This yields $\nabla\phi(x, t_0) = \frac{x}{t_0}$ under the assumption $\nabla\phi(0, t_0) = 0$. Thus, up to a constant, $\phi(x, t_0) = \frac{\|x\|^2}{2t_0}$. By the Hopf-Lax formula for the solution to the Hamilton-Jacobi equation (2) with $\phi(x, t_0) = \frac{\|x\|^2}{2t_0}$, we have

$$\phi(x, t) = \inf_{y \in \mathbb{R}^n} \left\{ \frac{\|x\|^2}{2t_0} + \frac{\|x - y\|^2}{2(t - t_0)} \right\} = \frac{\|x\|^2}{2t}, \quad \forall t > t_0, x \in \mathbb{R}^n.$$

By the uniqueness of the smooth solution to the Hamilton-Jacobi equation (2), we see that $\phi(x, t) = \frac{\|x\|^2}{2t}$ for all $t > 0$. Solving the transport equation (1) with the initial data $\lim_{t \rightarrow 0} \rho(t, x) = \delta_0(x)$, we have

$$\rho(x, t) = \frac{e^{-\frac{\|x\|^2}{4t^2}}}{(4\pi t^2)^{n/2}}, \quad \forall t > 0, x \in \mathbb{R}^n.$$

This finishes the proof of Theorem 4.7. □

4.5 Comparison between Theorem 1.1 and Theorem 1.3

Theorem 1.1 and Theorem 1.3 have some similar feature. Their proofs are also quiet similar. Both of them are based on the dissipation formulas of the first order and the second order derivatives of the Boltzmann entropy along the backward heat equation (i.e., the backward gradient flow of the Boltzmann-Shannon entropy) and the geodesic flow on the Wasserstein space $P_2(M, \mu)$. This lead us to the following observation: On compact Riemannian manifolds (M, g) with weighted measure $d\mu = e^{-f}dv$, if the $CD(0, m)$ -condition holds, i.e., $Ric_{m,n}(L) \geq 0$, then the relative Boltzmann-Shannon entropy $H_m(\rho, t) = \text{Ent}(\rho(t)) - \text{Ent}(\rho_m(t))$ is convex along the geodesic flow on the Wasserstein space $P_2^\infty(M, \mu)$, and the relative Boltzmann-Shannon entropy $H_m(u, t) = \text{Ent}(u(t)) - \text{Ent}(u_m(t))$ is convex along the backward gradient flow of $\text{Ent}(u) = \int_M u \log u d\mu$ on the Wasserstein space $P_2^\infty(M, \mu)$. This leads us to raise the question whether there is an essential reason for which the Boltzmann-Shannon entropy share the convexity property along the geodesic flow and the gradient flow on $P_2^\infty(M, \mu)$.

On the other hand, there is a difference between the W -entropy formula for the heat equation of the Witten Laplacian on Riemmanian manifold and the W -entropy formula

for the geodesic flow on the Wasserstein space, i.e., $\frac{g}{2t}$ appears in (5), while $\frac{g}{t}$ appears in (4), and their rigidity models are also different (see Theorem 3.4 and Theorem 4.7). An intuitive interpretation for this difference can be given as follows. The heat kernel (i.e., the fundamental solution of the heat equation) of the Laplacian on \mathbb{R}^m is the transition probability of Brownian motion starting from time 0 to time t . The variance of the distance that the “Brownian particle” moving along its trajectory (i.e., Brownian motion) B_t on \mathbb{R}^m during the time interval $[0, t]$ is given by $\mathbb{E}[|B_t|^2] = mt$. While the transport equation (1) and the Hamilton-Jacobi equation (2) describe the motion of the “light particle” along the geodesic on the Wasserstein space. Thus, assuming that the velocity of the “light particle” has the unit speed along each direction, then the distance (denoted by $|X_t|$) of the “light particle” moving along the geodesic during time interval $[0, t]$ is indeed $|X_t| = t$. Hence, the “variance” of the distance of the “light particle” moving along the geodesic during time interval $[0, t]$ is $\mathbb{E}[|X_t|^2] = t^2$. This explains intuitively why the rigidity model for the W -entropy for the heat equation of the Witten Laplacian on complete Riemannian manifolds with $CD(0, m)$ -condition is the Gaussian space $(\mathbb{R}^m, g_0, N(0, t\text{Id}))$, while the rigidity model for the W -entropy for the geodesic flow on the Wasserstein space over complete Riemannian manifolds with $CD(0, m)$ -condition is the Gaussian space $(\mathbb{R}^m, g_0, N(0, t^2\text{Id}))$. Here g_0 denotes the Euclidean metric on \mathbb{R}^m , Id is the unit matrix on \mathbb{R}^m , and $N(0, t\text{Id})$ denotes the Gaussian distribution on \mathbb{R}^m with mean zero and variance $t\text{Id}$.

5 Langevin deformation on finite dimensional manifolds

In this section we introduce the Langevin deformation of geometric flows on the cotangent bundle T^*M over a complete Riemannian manifold (M, g) , which interpolates the geodesic flow on T^*M and the gradient flow of a potential function on M . Our work has been inspired by Bismut [3, 4] who introduced a deformation of hypoelliptic Laplacians on T^*M , which is the infinitesimal generator of the Langevin diffusion process interpolating the geodesic flow on T^*M and the Brownian motion on M . The ideas and results in this section will be extended in Section 6 to the infinite dimensional Wasserstein space over compact Riemannian manifolds.

5.1 The idea for construction of the Langevin deformation of flows

We first describe J.-M. Bismut’s idea for the construction of a family of hypoelliptic Laplacians on the cotangent bundle over Riemannian manifolds ([3, 4]). Let $c > 0$ be a parameter, let (x_t, v_t) be the Langevin diffusion process on the tangent bundle TM over a complete Riemannian manifold M which solves the following stochastic differential equation

$$\dot{x} = \frac{v}{c}, \tag{47}$$

$$dv = -\frac{v}{c^2}dt + \frac{dw_t}{c}. \tag{48}$$

where dw_t denotes the Itô differential of Brownian motion w_t on M . This is the stochastic differential equation for the Langevin hypoelliptic diffusion process (x_t, v_t) on the tangent bundle TM over M . The position process x_t satisfies the second order stochastic differential

equation

$$c^2 \ddot{x} = -\dot{x} + \dot{w}_t, \quad (49)$$

where \dot{w}_t denotes the formal Itô or Stratonovich derivation of Brownian path w_t on M . As was pointed out by Bismut [3, 4], taking $c \rightarrow 0$, the limiting process x_t is the Brownian motion on M , ie.,

$$\dot{x} = \dot{w}_t,$$

and when $c \rightarrow \infty$, to make sense the Langevin stochastic differential equation (49), the limiting process x_t must satisfy the geodesic equation

$$\ddot{x} = 0.$$

Thus the Langevin diffusion processes (x_t, v_t) provide a deformation of geometric flows which interpolate the geodesic flows $\ddot{x} = 0$ on the cotangent bundle T^*M over M and the Brownian motion $x_t = w_t$ on the underlying Riemannian manifold M .

Let V be a smooth function on M . Instead of introducing the above Langevin diffusion processes on TM , let us introduce the following deformation of geometric flows on TM

$$\dot{x} = \frac{v}{c}, \quad (50)$$

$$\dot{v} = -\frac{v}{c^2} + \frac{\nabla V(x)}{c}. \quad (51)$$

Then x_t satisfies the second order ordinary differential equation

$$c^2 \ddot{x} = -\dot{x} + \nabla V(x). \quad (52)$$

The equation (52) is indeed the Newton-Langevin equation which describes the motion of particles moving in a fluid with friction coefficient c^{-2} and with an external potential $c^{-2}V$. It defines a family of geometric flows on T^*M which interpolates the geodesic flow $\ddot{X} = 0$ and the backward gradient flow $\dot{X} = \nabla V(X)$. Indeed, similarly to Bismut's situation, when $c \rightarrow 0$, the limiting flow x_t is the backward gradient flow of V , i.e.,

$$\dot{x}_t = \nabla V(x_t),$$

and when $c \rightarrow \infty$, to make sense the Newton-Langevin equation (52), the limiting flow x_t must satisfy the geodesic equation

$$\ddot{x} = 0.$$

In view of this, (x_t, v_t) is a deformation (called the Langevin deformation) of geometric flows on TM which interpolate the geodesic flows $\ddot{x} = 0$ on the cotangent bundle T^*M over M and the gradient flow $\dot{x}_t = -\nabla V(x_t)$ on the underlying Riemannian manifold M .

Following Bismut [3, 4] and Villani [37], we can use a Hamiltonian point of view to give an interpretation of the Langevin deformation of flows. Let

$$H(x, v) = \frac{|v|^2}{2c} + \frac{V(x)}{c} \quad (53)$$

be the Hamiltonian energy of a particle moving in the cotangent bundle, where V is the external potential. Then

$$\nabla H(x, v) = \left(\frac{\partial H}{\partial x}, \frac{\partial H}{\partial v} \right)^\tau = (c^{-1} \nabla V(x), c^{-1} v)^\tau.$$

Let

$$A = \begin{pmatrix} 0 & I \\ I & -c^{-1}I \end{pmatrix}.$$

Then

$$A \nabla H(x, v) = (c^{-1} v, -c^{-2} v + c^{-1} \nabla V(x))^\tau.$$

Thus, (x_t, v_t) can be regarded as the “ A -Hamiltonian flow” defined by (Villani [37])

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = A \nabla H(x, v). \quad (54)$$

5.2 Variational formula for the Hamiltonian functional

In this subsection we prove the variational formula of the Hamiltonian energy and the external potential along the Langevin deformation of geometric flows which introduced in above subsection.

Theorem 5.1 *For any $c > 0$, let (x_t, v_t) be the deformation of flows on T^*M defined by (50) and (51). Let*

$$H(x, v) = \frac{|v|^2}{2} + V(x).$$

Then

$$\frac{d^2}{dt^2} H(x_t, v_t) = 2 |\dot{v}|^2 + 2 \nabla^2 V(x) \left(\frac{v}{c}, \frac{v}{c} \right).$$

In particular, if V is K -convex, i.e., $\nabla^2 V \geq K$, then

$$\frac{d^2}{dt^2} H(x_t, v_t) \geq 2K |\dot{x}|^2 + 2c^2 |\ddot{x}|^2.$$

Proof. By simple calculation, we have

$$\begin{aligned} \frac{d}{dt} H(x, v) &= v \cdot \dot{v} + \nabla V \cdot \dot{x} \\ &= v \cdot \left(-\frac{v}{c^2} + \frac{\nabla V}{c} \right) + \nabla V \cdot \frac{v}{c} \\ &= -\frac{|v|^2}{c^2} + 2 \nabla V \cdot \frac{v}{c}. \end{aligned}$$

Therefore

$$\begin{aligned}
\frac{d^2}{dt^2}H(x, v) &= -\frac{2}{c^2}v \cdot \dot{v} + 2\nabla^2 V\left(\frac{v}{c}, \frac{v}{c}\right) + 2\nabla V \cdot \frac{\dot{v}}{c} \\
&= -2\dot{v} \left(\frac{v}{c^2} - \frac{\nabla V}{c}\right) + 2\nabla^2 V\left(\frac{v}{c}, \frac{v}{c}\right) \\
&= 2\left|\frac{v}{c^2} - \frac{\nabla V}{c}\right|^2 + 2\nabla^2 V\left(\frac{v}{c}, \frac{v}{c}\right) \\
&= 2|\dot{v}|^2 + 2\nabla^2 V(\dot{x}, \dot{x}).
\end{aligned}$$

This finishes the proof of Theorem 5.1. \square

5.3 W -entropy formula for Langevin deformation on T^*M

In this subsection, we introduce a variant of the W -entropy functional and to prove its monotonicity along the Langevin deformation of geometric flows (x_t, v_t) on T^*M . Our first observation is the following

Proposition 5.2 *For any $c > 0$, let (x_t, v_t) be the deformed flow on T^*M defined by (50) and (51). Then*

$$\begin{aligned}
\left(\frac{d^2}{dt^2} + \frac{1}{c^2} \frac{d}{dt}\right)V(x) &= \frac{1}{c^2} [\nabla^2 V(v, v) + |\nabla V|^2], \\
\left(\frac{d^2}{dt^2} + \frac{2}{c^2} \frac{d}{dt}\right)H(x, v) &= \frac{2}{c^2} [\nabla^2 V(v, v) + |\nabla V|^2].
\end{aligned}$$

In particular, if $\nabla^2 \geq K$, where $K \in \mathbb{R}$ is a constant, we have

$$\begin{aligned}
\left(\frac{d^2}{dt^2} + \frac{1}{c^2} \frac{d}{dt}\right)V(x) &\geq \frac{1}{c^2} [K|v|^2 + |\nabla V|^2], \\
\left(\frac{d^2}{dt^2} + \frac{2}{c^2} \frac{d}{dt}\right)H(x, v) &\geq \frac{2}{c^2} [K|v|^2 + |\nabla V|^2].
\end{aligned}$$

Proof. Indeed, a simple calculation yields

$$\begin{aligned}
\frac{d}{dt}V(x) &= \nabla V(x) \cdot \dot{x} = \nabla V(x) \cdot \frac{v}{c}, \\
\frac{d^2}{dt^2}V(x) &= \nabla^2 V(\dot{x}, \dot{x}) + \nabla V \cdot \ddot{x} \\
&= \frac{1}{c^2} \nabla^2 V(v, v) + \frac{1}{c^2} \nabla V \cdot (-\dot{x} + \nabla V).
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{d^2}{dt^2}V(x) + \frac{1}{c^2} \frac{d}{dt}V(x) &= \frac{1}{c^2} \nabla^2 V(v, v) + \frac{1}{c^2} \nabla V \cdot (-\dot{x} + \nabla V) + \frac{1}{c^2} \nabla V(x) \cdot \dot{x} \\
&= \frac{1}{c^2} [\nabla^2 V(v, v) + |\nabla V|^2].
\end{aligned}$$

Similarly, we have

$$\begin{aligned} \left(\frac{d^2}{dt^2} + \frac{2}{c^2} \frac{d}{dt} \right) H &= 2 \left| \frac{v}{c^2} - \frac{\nabla V}{c} \right|^2 + 2 \nabla^2 V \left(\frac{v}{c}, \frac{v}{c} \right) + \frac{2}{c^2} \left(-\frac{|v|^2}{c^2} + 2 \nabla V \cdot \frac{v}{c} \right) \\ &= \frac{2}{c^2} [|\nabla V|^2 + \nabla^2 V(v, v)]. \end{aligned}$$

This finishes the proof. \square

The variational formulas in Proposition 5.2 lead us to introduce a variant of Perelman's W -entropy functional and to prove its monotonicity along the deformed flows (x_t, v_t) on T^*M . To describe the general idea how to introduce the W -entropy type functional, let $\{\gamma(t), t \in [0, T]\}$ be a smooth curve on the cotangent bundle T^*M over a complete Riemannian manifold M , and suppose that we have already proved the inequality

$$\frac{d^2}{dt^2} H(\gamma(t)) + \beta(t) \frac{d}{dt} H(\gamma(t)) \geq 0, \quad t \in [0, T], \quad (55)$$

where $\beta : [0, T] \rightarrow \mathbb{R}$ is a suitable function. Let $\alpha : [0, T] \rightarrow \mathbb{R}$ be a smooth function and define the W -entropy by the revised Boltzmann entropy formula in statistical mechanics⁴

$$W_H(\gamma(t)) := H(\gamma(t)) + \alpha(t) \frac{d}{dt} H(\gamma(t)). \quad (56)$$

Then

$$\frac{d}{dt} W_H(\gamma(t)) = \alpha(t) \left[\frac{d^2}{dt^2} H(\gamma(t)) + \frac{1 + \dot{\alpha}}{\alpha} \frac{d}{dt} H(\gamma(t)) \right]$$

In particular, choosing $\alpha(t)$ such that it solves the ODE

$$\frac{1 + \dot{\alpha}}{\alpha} = \beta, \quad t \in [0, T],$$

we have

$$\frac{d}{dt} W_H(\gamma(t)) = \alpha(t) \left[\frac{d^2}{dt^2} H(\gamma(t)) + \beta(t) \frac{d}{dt} H(\gamma(t)) \right]. \quad (57)$$

In particular, $\frac{d}{dt} W_H(\gamma(t))$ has a sign provided that $\alpha(t) \geq 0$ or $\alpha(t) \leq 0$ on $[0, T]$. This idea was first used in our previous paper [16] when we introduced the W -entropy functional and proved its variational formula along the heat equation of the Witten Laplacian $\partial_t u = Lu$ with $\alpha(t) = \frac{\sinh(2Kt)}{2K}$ and $\beta(t) = 2K \coth(Kt)$ on complete Riemannian manifolds with the $CD(K, \infty)$ -condition. See Theorem 1.8 in [16].

Now we introduce W -entropy and prove its variational formula for the Langevin deformation of geometric flows on T^*M .

Theorem 5.3 *For any $c > 0$, let (x_t, v_t) be the deformation of flows on T^*M defined by (50) and (51). Define*

$$\begin{aligned} W_{H,c}(x, v) &= H(x, v) + \frac{c^2(1 - e^{-\frac{t}{c^2}})}{2} \frac{d}{dt} H(x, v), \\ W_{V,c}(x) &= V(x) + c^2(1 - e^{-\frac{t}{c^2}}) \frac{d}{dt} V(x). \end{aligned}$$

⁴The Boltzmann entropy formula in statistical mechanics corresponds to the special case $\alpha(t) = t$. In [29], Perelman used the Boltzmann entropy formula to introduce his W -entropy functional and proved its monotonicity along the conjugate heat equation of the Ricci flow.

Then

$$\begin{aligned}\frac{d}{dt}W_{H,c}(x, v) &= (1 - e^{\frac{2t}{c^2}}) [\nabla^2 V(v, v) + |\nabla V|^2], \\ \frac{d}{dt}W_{V,c}(x, v) &= (1 - e^{\frac{t}{c^2}}) [\nabla^2 V(v, v) + |\nabla V|^2].\end{aligned}$$

In particular, if $\nabla^2 V \geq 0$, then for all $c > 0$, we have

$$\frac{d}{dt}W_{H,c}(x, v) \leq 0, \quad \forall t \geq 0,$$

and

$$\frac{d}{dt}W_{V,c}(x) \leq 0, \quad \forall t \geq 0.$$

Proof. By Corollary 5.2, we have $\beta(t) = \frac{2}{c^2}$. Solving the ODE

$$\frac{1 + \dot{\alpha}}{\alpha} = \frac{2}{c^2},$$

we have

$$\alpha(t) = \frac{c^2}{2} \left(1 - e^{\frac{2t}{c^2}}\right).$$

Thus

$$\begin{aligned}\frac{d}{dt}W_{H,c}(x, v) &= \frac{c^2(1 - e^{\frac{2t}{c^2}})}{2} \left(\frac{d^2}{dt^2} + \frac{2}{c^2} \frac{d}{dt} \right) H(x, v) \\ &= (1 - e^{\frac{2t}{c^2}}) [\nabla^2 V(v, v) + |\nabla V|^2].\end{aligned}$$

In particular, if $\nabla^2 V \geq 0$, then for all $t \geq 0$, we have

$$\frac{d}{dt}W_{H,c}(x, v) \leq 0.$$

Similarly, we can prove the corresponding result for $W_{V,c}(x, v)$. □

6 Langevin deformation on Wasserstein space and compressible Euler equation with damping

In this section we extend the idea of Section 5 to introduce the Langevin deformation of geometric flows on the infinite dimensional Wasserstein space over compact Riemannian manifolds. We observe that the Langevin deformation of geometric flows is indeed the potential flow of the compressible Euler equation with damping on manifolds. Using the Kato-Majada theory of quasi-linear symmetric hyperbolic system we prove the existence, uniqueness and regularity of solution to the Cauchy problem of the compressible Euler equation with damping on compact Riemannian manifolds. Finally we prove the existence, uniqueness and regularity of solution to the Cauchy problem of the Langevin deformation of flows on the Wasserstein space over compact Riemannian manifolds.

6.1 Langevin deformation on $T^*P_2^\infty(M, \mu)$

Let M be a compact Riemannian manifold, $P_2^\infty(M, \mu)$ the smooth Wasserstein space over M equipped with the weighted volume measure $d\mu = e^{-f} dv$. Let $V : P_2^\infty(M, \mu) \rightarrow \mathbb{R}$ be a smooth function. Extending the idea in Section 5, we introduce the following ODEs on $T^*P_2^\infty(M, \mu)$ as follows

$$\begin{aligned}\partial_t \rho &= \frac{v}{c}, \\ \partial_t v &= -\frac{v}{c^2} + \frac{\nabla V(\rho)}{c},\end{aligned}$$

where $\rho d\mu : [0, T] \rightarrow P_2^\infty(M, \mu)$ is a smooth curve. Since $\dot{\rho} d\mu \in T_{\rho d\mu} P_2^\infty(M, \mu)$, there exists a function ϕ on M such that

$$\partial_t \rho + \nabla_\mu^*(\rho \nabla \phi) = 0. \quad (58)$$

Thus the first equation reads as

$$v = -c \nabla_\mu^*(\rho \nabla \phi),$$

and the second equation can be written as

$$c^2 \nabla_{\dot{\rho}} \dot{\rho} = -\dot{\rho} + \nabla V(\rho).$$

According to Otto [24], we have

$$\nabla V(\rho) = -\nabla_\mu^* \left(\rho \nabla \frac{\delta V}{\delta \rho} \right),$$

where $\frac{\delta V}{\delta \rho}$ denotes the L^2 -derivative of V with respect to ρ . Moreover, according to Lott [18], we have

$$\nabla_{\dot{\rho}} \dot{\rho} = -\nabla_\mu^* \left(\rho \nabla \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 \right) \right). \quad (59)$$

Thus, up to an additional constant for ϕ , the Langevin deformation of geometric flows on $P_2^\infty(M, \mu)$ is given as follows

$$\partial_t \rho + \nabla_\mu^*(\rho \nabla \phi) = 0, \quad (60)$$

$$c^2 \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 \right) = -\phi + \frac{\delta V}{\delta \rho}. \quad (61)$$

Note that, when $c = 0$, Eq. (61) reads

$$\phi = \frac{\delta V}{\delta \rho},$$

which yields that ρ is the backward gradient flow of V on the Wasserstein space $P_2^\infty(M, \mu)$

$$\partial_t \rho = -\nabla_\mu^* \left(\rho \nabla \frac{\delta V}{\delta \rho} \right). \quad (62)$$

When $c = \infty$, to make sense Eq. (60) and Eq. (61), we have

$$\partial_t \rho + \nabla_\mu^*(\rho \nabla \phi) = 0, \quad (63)$$

$$\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 = 0, \quad (64)$$

which is indeed the geodesic flow on the cotangent bundle over the Wasserstein space $P_2^\infty(M, \mu)$. In view of this, the Langevin deformation of flows can be regarded as an interpretation between the gradient flow on the Wasserstein space $P_2^\infty(M, \mu)$ and the geodesic flow on the cotangent bundle over $P_2^\infty(M, \mu)$.

6.2 Compressible Euler equation with damping

Let $u = \nabla \phi$, $\gamma = \frac{1}{c^2}$, and $p \in C^1(\mathbb{R})$ be such that $\nabla p(\rho) = \rho \frac{\delta V}{\delta \rho}(\rho)$. Then, if (ρ, ϕ) satisfies the transport equation (11) (i.e., (60)) and the deformed Hamilton-Jacobi equation (12) (i.e., (61)), we have

$$\partial_t \rho + \nabla_\mu^*(\rho u) = 0, \quad (65)$$

$$\partial_t u + u \cdot \nabla u = -\gamma u - \frac{1}{c^2} \frac{\nabla p(\rho)}{\rho}. \quad (66)$$

In the case $M = \mathbb{R}^n$ and $\mu = dx$, the above system is indeed the compressible Euler equation with damping in the isentropic fluid, where ρ is the density of fluid, u is the velocity of the fluid, and $\gamma = \frac{1}{c^2}$ is the friction constant, $p(\rho)$ is the pressure of the fluid.

In the case M is a compact Riemannian manifold, we can regard (65) and (66) as the compressible Euler equation with damping in the isentropic fluid on the compact Riemannian manifold (M, g) equipped with the reference measure μ . In this case, the compressible Euler equation with damping can be rewritten as follows

$$\partial_t \rho + \nabla_\mu^*(\rho u) = 0, \quad (67)$$

$$\partial_t u + \nabla_u u = -\gamma u - \frac{1}{c^2} \nabla V'(\rho). \quad (68)$$

where $\nabla_u u$ denotes the Levi-Civita covariant derivative of the vector field u along the velocity field u of the trajectory of the fluid. By identifying u with its dual u^* , and identifying $\nabla V'(\rho)$ with its dual $dV'(\rho)$, with respect to the Riemannian metric on M , we can rewrite (68) as follows

$$\partial_t u^* + \nabla_u u^* = -\gamma u^* - \frac{1}{c^2} dV'(\rho). \quad (69)$$

When $\gamma = 0$, the compressible Euler equations and the deformed Hamilton-Jacobi equation with the transport equation have been well studied in the literature at least in the Euclidean case. See e.g. Carles [8] and reference therein. The case of compact Riemannian manifolds is as the same as in the Euclidean case. On the other hand, the compressible Euler equation with damping on Euclidean space has been also well studied in the literature. See Wang and Yang [39]. See also Sideris, Thomases and Wang [30] and reference therein. In this subsection, we develop a little bit more detail on the link between the deformed

Hamilton-Jacobi equation and the compressible Euler equation with damping on compact Riemannian manifolds.

Recall the Sobolev inequalities on compact Riemannian manifolds. We only consider the unweighted case $\mu = \nu$. The general case can be treated similarly. By [1], there exists a constant $C_{\text{Sob}} > 0$ such that

$$\|f\|_{\frac{2n}{n-2}} \leq C_{\text{Sob}}(\|\nabla f\|_2 + \|f\|_2), \quad \forall f \in C^\infty(M).$$

Moreover, for any $\alpha \in (0, 1)$, if $k > \alpha + \frac{n}{2}$, the Kondrakov embedding theorem holds

$$\|f\|_{C^{0,\alpha}} \leq C_\alpha \|f\|_{k,2}.$$

In particular, we have

$$\|f\|_\infty \leq C_\alpha \|f\|_{k,2}. \quad (70)$$

Let $H^s(M)$ denotes the Sobolev space equipped with the Sobolev norm

$$\|f\|_{s,2} = \left(\sum_{|\alpha| \leq s} \|D^\alpha f\|_{L^2}^2 \right)^{1/2}$$

where α is a multi-index, and for any vector valued function $U = (u_1, \dots, u_n)$ on M ,

$$\|U\|_{s,2} = \sum_{i=1}^n \|u_i\|_{s,2}$$

For any $s \in \mathbb{Z}$, we define the Banach space

$$X_s = \{(f, g) | f \in H^2(M), g \in \oplus^n H^s(M)\}$$

equipped with the norm $\|(f, g)\|_{s,2} = \|f\|_{s,2} + \|g\|_{s,2}$.

The following result gives the local existence and uniqueness of the solution to the compressible Euler equation with damping on compact Riemannian manifolds.

Theorem 6.1 (*Local existence and uniqueness of smooth solution*) *Let M be \mathbb{R}^n or a compact Riemannian manifold, $s = [\frac{n}{2}] + 1$. Let $V(\rho) = \int_M \rho \log \rho d\mu$ or $V(\rho) = \frac{1}{m-1} \int_M \rho^m d\mu$ for $m > 1$. Suppose that $(\rho_0, u_0) \in H^{s+1}(M)$ with $\rho_0 > 0$. Then, there exists a constant $T > 0$ such that the Cauchy problem of the compressible Euler with damping (65) and (66) has a unique smooth solution (ρ, u) in $C([0, T], H^{s+1}(M)) \times H^s(M)$.*

Proof. This is a standard result which can be proved by the method in Kato [10] and Majda [20]. \square

The following result gives the global existence and uniqueness of the solution to the compressible Euler equation with damping on compact Riemannian manifolds.

Theorem 6.2 (*Global existence and uniqueness of smooth solution with small initial data*) *Let M be \mathbb{R}^n or a compact Riemannian manifold. Let $V(\rho) = \int_M \rho \log \rho d\mu$ or $V(\rho) = \frac{1}{m-1} \int_M \rho^m d\mu$ for $m > 1$. Let $s = [\frac{n}{2}] + 1, l \geq 2$. Then there exists $\delta_0 > 0$ such that if $(\rho_0 -$*

$1, u_0) \in H^{s+l}(M)$ is a small smooth initial value in the sense that $\|\rho_0 - 1\|_{s+l,2} + \|u_0\|_{s+l,2} \leq \delta_0$ is sufficiently small. Then the Cauchy problem of the compressible Euler equation with damping (65) and (66) admits a unique global smooth solution $(\rho, u) \in C([0, \infty), H^{s+l}(M) \times H^{s+l-1}(M))$ with initial value (ρ_0, u_0) and satisfying the following energy estimate

$$\begin{aligned} & \|\partial_t \rho(t)\|_{s+l-1,2}^2 + \|\rho(t) - 1\|_{s+l,2}^2 + \|u(t)\|_{s+l,2}^2 \\ & + \int_0^t (\|\partial_t \rho(r)\|_{s+l-1,2} + \|\nabla \rho(r)\|_{s+l-1,2}^2 + \|u(r)\|_{s+l,2}^2) dr \\ & \leq C(\|\partial_t \rho(0)\|_{s+l-1,2} + \|\rho_0 - 1\|_{s+l,2} + \|u_0\|_{s+l,2}). \end{aligned} \quad (71)$$

Proof. In the case $M = \mathbb{R}^n$, $V(\rho) = \int_M \rho \log \rho d\mu$ or $V(\rho) = \frac{1}{m-1} \int_M \rho^m d\mu$ with $m > 1$, this is the well-established result due to Wang and Yang [39]. See also Sideris, Thomases and Wang [30] for the case $M = \mathbb{R}^n$ and $V(\rho) = \frac{1}{m-1} \int_M \rho^m d\mu$ with $m > 1$. In the case M is a compact Riemannian manifold and $V(\rho) = \frac{1}{m-1} \int_M \rho^m d\mu$ with $m > 1$, the proof of theorem is similar to the ones in [39, 30]. In the case M is a compact Riemannian manifold and $V(\rho) = \int_M \rho \log \rho d\mu$, we can modify the proof of the main results in [39, 30]. The main point here is that on compact Riemannian manifold, the positivity of the initial data $\rho_0 > 0$ implies that there exists a constant $\varepsilon_0 > 0$ such that $\rho_0 \geq \varepsilon_0 > 0$, and the argument used in Wang and Yang [39] can be extended to the case $V(\rho) = \int_M \rho \log \rho d\mu$ on compact Riemannian manifolds. We can also modify the argument used in Sideris, Thomases and Wang [30] by taking the sound speed to be $\sigma(\rho) = \log \rho$. To save the length of the paper, we omit the detail of the proof. \square

Let (e_i) be an ONB and normal at $x \in M$, writing $u = \sum_{i=1}^n u_i e_i$ and $u^* = \sum_{i=1}^n u_i e_i^*$, we can prove that the following formula holds

$$d\nabla_u u^* = \nabla_u(du^*) + \sum_{i=1}^n du_i \wedge \nabla_{e_i} u^* + \sum_{k=1}^n e_k^* \wedge R(e_k, u)u^*. \quad (72)$$

Indeed, since (e_i) is an ONB and normal at $x \in M$, we have

$$\begin{aligned} d\nabla_u u^* &= \sum_{k=1}^n e_k^* \wedge \nabla_{e_k}(\nabla_u u^*) \\ &= \sum_{k=1}^n e_k^* \wedge \nabla_{e_k} \left(\sum_{i=1}^n u_i \nabla_{e_i} u^* \right) \\ &= \sum_{k,i=1}^n (e_k^* \wedge \nabla_{e_k} u_i \nabla_{e_i} u^* + u_i e_k^* \wedge \nabla_{e_k} \nabla_{e_i} u^*) \\ &= \sum_{i=1}^n du_i \wedge \nabla_{e_i} u^* + \sum_{k,i=1}^n u_i e_k^* \wedge \nabla_{e_k} \nabla_{e_i} u^*, \end{aligned}$$

and

$$\begin{aligned}
\nabla_u du^* &= \sum_{i=1}^n u_i \nabla_{e_i} (du^*) \\
&= \sum_{i=1}^n u_i \nabla_{e_i} \left(\sum_{k=1}^n e_k^* \wedge \nabla_{e_k} u^* \right) \\
&= \sum_{k,i=1}^n (u_i \nabla_{e_i} e_k^* \wedge \nabla_{e_k} u^* + u_i e_k^* \wedge \nabla_{e_i} \nabla_{e_k} u^*) \\
&= \sum_{k,i=1}^n u_i e_k^* \wedge \nabla_{e_i} \nabla_{e_k} u^*.
\end{aligned}$$

Hence

$$\begin{aligned}
d\nabla_u u^* - \nabla_u (du^*) &= \sum_{i=1}^n du_i \wedge \nabla_{e_i} u^* + \sum_{k,i=1}^n u_i e_k^* \wedge \nabla_{e_k} \nabla_{e_i} u^* - \sum_{k,i=1}^n u_i e_k^* \wedge \nabla_{e_i} \nabla_{e_k} u^* \\
&= \sum_{i=1}^n du_i \wedge \nabla_{e_i} u^* + \sum_{k,i=1}^n u_i e_k^* \wedge R(e_k, e_i) u^*.
\end{aligned}$$

This proves (72). Note that

$$\sum_{k=1}^n e_k^* \wedge R(e_k, u) u^* \in \Gamma(\Lambda^2 T^* M).$$

Taking exterior differentiation on the both sides of the compressible Euler equation (69), letting $\omega = du^* = \sum_{i=1}^n du_i \wedge e_i^* \in \Gamma(\Lambda^2 T^* M)$, and using $ddV'(\rho) = 0$, we have

$$\partial_t \omega + \nabla_u \omega + \sum_{i=1}^n (\omega(e_i)) \wedge \nabla_{e_i} u^* + \sum_{k=1}^n e_k^* \wedge R(e_k, u) u^* = -\gamma \omega. \quad (73)$$

We now prove the following

Claim $\sum_{k=1}^n e_k^* \wedge R(e_k, u) u^* = 0$ for all $u \in \Gamma(TM)$.

Indeed, for any $i, j = 1, \dots, n$, acting on (e_i, e_j) , we have

$$\begin{aligned}
&\sum_{k=1}^n (e_k^* \wedge R(e_k, u) u^*)(e_i, e_j) \\
&= \sum_{k=1}^n e_k^*(e_i) (R(e_k, u) u^*)(e_j) - e_k^*(e_j) (R(e_k, u) u^*)(e_i) \\
&= (R(e_i, u) u^*)(e_j) - (R(e_j, u) u^*)(e_i).
\end{aligned}$$

Note that, for any one-form α and vector fields X and Y , $\nabla_X \alpha(Y) = (\nabla_X \alpha)(Y) - \alpha(\nabla_X Y)$. Taking $\alpha = \nabla_u u^*$, we have

$$\begin{aligned}
\nabla_{e_i} ((\nabla_u u^*)(e_j)) &= (\nabla_{e_i} \nabla_u u^*)(e_j) - (\nabla_u u^*)(\nabla_{e_i} e_j) \\
&= (\nabla_{e_i} \nabla_u u^*)(e_j),
\end{aligned}$$

and

$$\nabla_{e_i}((\nabla_u u^*)(e_j)) = \nabla_{e_i}(\nabla_u u^*(e_j) + u^*(\nabla_u e_j)) = \nabla_{e_i} \nabla_u u_j + \nabla_{e_i}(u^*(\nabla_u e_j)).$$

This implies

$$(\nabla_{e_i} \nabla_u u^*)(e_j) = \nabla_{e_i} \nabla_u u_j + \nabla_{e_i}(u^*(\nabla_u e_j)).$$

On the other hand,

$$\begin{aligned} (\nabla_u \nabla_{e_i} u^*)(e_j) &= \nabla_u((\nabla_{e_i} u^*)(e_j)) + (\nabla_{e_j} u^*)(\nabla_u e_j) \\ &= \nabla_u(\nabla_{e_i}(u^*(e_j)) + u^*(\nabla_{e_i} e_j)) \\ &= \nabla_u \nabla_{e_i} u_j + \nabla_u(u^*(\nabla_{e_i} e_j)). \end{aligned}$$

Hence

$$\begin{aligned} (R(e_i, u)u^*)(e_j) &= \nabla_{e_i}(u^*(\nabla_u e_j)) - \nabla_u(u^*(\nabla_{e_i} e_j)) \\ &= \nabla_{e_i}\langle u, \nabla_u e_j \rangle - \nabla_u\langle u, \nabla_{e_i} e_j \rangle \\ &= \langle \nabla_{e_i} u, \nabla_u e_j \rangle + \langle u, \nabla_{e_i} \nabla_u e_j \rangle - \langle \nabla_u u, \nabla_{e_i} e_j \rangle - \langle u, \nabla_u \nabla_{e_i} e_j \rangle \\ &= \langle u, \nabla_{e_i} \nabla_u e_j \rangle - \langle u, \nabla_u \nabla_{e_i} e_j \rangle \\ &= \langle u, R(e_i, u)e_j \rangle. \end{aligned}$$

Exchanging e_i and e_j , we have

$$(R(e_j, u)u^*)(e_i) = \langle u, R(e_j, u)e_i \rangle.$$

Therefore

$$\begin{aligned} (R(e_i, u)u^*)(e_j) - (R(e_j, u)u^*)(e_i) &= \langle u, R(e_i, u)e_j \rangle - \langle u, R(e_j, u)e_i \rangle \\ &= R(e_i, u, e_j, u) - R(e_j, u, e_i, u). \end{aligned}$$

Note that $R(X, Y, Z, W) = R(Z, W, X, Y)$. Thus

$$R(e_i, u, e_j, u) = R(e_i, u, e_i, u).$$

This yields

$$\sum_{k=1}^n (e_k^* \wedge R(e_k, u)u^*)(e_i, e_j) = 0.$$

The **Claim** is proved.

Thus, Eq. (73) reads as follows

$$\partial_t \omega + \nabla_u \omega + \sum_{i=1}^n (\omega(e_i)) \wedge \nabla_{e_i} u^* = -\gamma \omega,$$

which will be written briefly as follows

$$\partial_t \omega + \nabla_u \omega + \omega \wedge \nabla u^* = -\gamma \omega.$$

Let (ρ, u) be the unique smooth solution to the compressible Euler equation with damping (65) and (66) on $[0, T] \times M$ with initial data (ρ_0, u_0) . By the Sobolev embedding inequality (70) and (71) in Theorem 6.2, there exists a constant $C > 0$ which depends on the Sobolev norms of the initial data (ρ_0, u_0) and the Sobolev constant on M such that

$$\sup_{t \in [0, T]} \|\nabla u(t, \cdot)\|_\infty \leq C. \quad (74)$$

We now prove the following result which has its own interest.

Theorem 6.3 ⁵ *Let $M = \mathbb{R}^n$ or a compact Riemannian manifold, (ρ, u) be a smooth solution to the compressible Euler equation with damping, i.e., (65) and (66). Let $\omega = du$. Suppose that $\frac{\nabla p(\rho)}{\rho} = \nabla V'(\rho)$. Then*

$$\partial_t \omega + u \cdot \nabla \omega + \omega \wedge \nabla u^* = -\gamma \omega. \quad (75)$$

Let C be the constant in (74). Then, for all $t \in [0, T]$, we have ⁶

$$\|\omega(t)\|_{L^p} \leq \|\omega(0)\|_{L^p} e^{(C-\gamma)t}. \quad (76)$$

In particular, if u_0 is a closed form, so is $u(t, \cdot)$, i.e., $du_0 = 0$ implies $du(t, \cdot) = 0$ on $[0, T]$.

Proof. We have already proved that ω satisfies Eq. (75). Taking inner product with $|\omega|^{p-2}\omega$ in the both sides of (75), and integrating on M , we have

$$\int_M \left\langle \frac{D}{\partial t} \omega, |\omega|^{p-2} \omega \right\rangle d\mu + \int_M \langle \omega \wedge \nabla u, |\omega|^{p-2} \omega \rangle d\mu = -\gamma \|\omega\|_p^p$$

where $\frac{D}{\partial t} \omega = \partial_t \omega + u \cdot \nabla \omega$. Note that

$$\left\langle \frac{D}{\partial t} \omega, |\omega|^{p-2} \omega \right\rangle = \frac{1}{p} \frac{D}{\partial t} |\omega|^p.$$

Hence

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \|\omega(t)\|_p^p &= - \int_M (\langle \omega \wedge \nabla u, |\omega|^{p-2} \omega \rangle d\mu - \gamma \|\omega(t)\|_p^p) d\mu \\ &\leq \int_M (|\nabla u| |\omega(t)|^p d\mu - \gamma \|\omega(t)\|_p^p) d\mu \\ &\leq \left(\sup_{t \in [0, T]} \|\nabla u\|_\infty - \gamma \right) \|\omega(t)\|_p^p \\ &\leq (C - \gamma) \|\omega(t)\|_p^p. \end{aligned}$$

By Gronwall inequality, we have

$$\|\omega(t)\|_p \leq \|\omega(0)\|_p \exp \{(C - \gamma) t\}.$$

Thus, if $\omega(0) = 0$, then for all $t \geq 0$, we have $\omega(t) = 0$. Hence, if $u^*(0, \cdot)$ is a closed one-form on M , then $u^*(t, \cdot)$ is also a closed one-form on M . □

We now state the main results of this section.

⁵When $M = \mathbb{R}^3$, $p = 2$ and $\omega = \nabla \times u$ is the vorticity of u , see Sideris, Thomases and Wang [30].

⁶In particular, if $\omega(0) = 0$ or the Sobolev norm of the initial data (ρ_0, u_0) is small enough such that $C < \gamma$, then (76) holds for all $t \in [0, \infty)$.

Theorem 6.4 *Let $M = \mathbb{R}^n$ or be a compact Riemannian manifold, $c \in [0, \infty]$. Given $(\rho_0, \phi_0) \in T^*P_2^\infty(M, \mu)$ with $\rho_0, \phi_0 \in C^\infty(M)$, there exists $T = T_c > 0$ such that the Cauchy problem of the transport equation (11) and the deformed Hamilton-Jacobi equation (12) has a unique solution $(\rho, \phi) \in C^1([0, T], C^\infty(M)^2)$.*

Proof. The cases $c = 0$ and $c = \infty$ are well known. For $c \in (0, \infty)$, consider the compressible Euler equation with damping on M

$$\partial_t u + u \cdot \nabla u = -\frac{u}{c^2} + \frac{1}{c^2} \frac{\delta V}{\delta \rho}, \quad u|_{t=0} = \nabla \phi_0, \quad (77)$$

$$\partial_t \rho + \nabla_\mu^*(\rho u) = 0, \quad \rho|_{t=0} = \rho_0. \quad (78)$$

By Theorem 6.1, if the initial data are in $H^s(M, \mu)$ for any $s > \frac{n}{2} + 1$, then there exists $T = T_c > 0$ such that above system has a unique solution $(\rho, u) \in C([0, T], H^s(M, \mu))^2$. Moreover, tame estimates show that the time of existence $T > 0$ can be chosen independent of $s > \frac{n}{2} + 1$.

By Theorem 6.3, if $u_0^* = d\phi_0$, $u^*(t, \cdot)$ is closed on M . For all $(t, x) \in [0, T] \times M$, let

$$\phi(t, x) = e^{-\gamma t} \phi_0(x) + e^{-\gamma t} \int_0^t e^{\gamma s} \left(f(\rho(s, x)) - \frac{1}{2} |u(s, x)|^2 \right) ds,$$

where $\gamma = \frac{1}{c^2}$ and $f(\rho) = \frac{1}{c^2} \frac{\delta V}{\delta \rho}$. We have

$$\partial_t \phi = -\gamma \phi + f(\rho(t, x)) - \frac{1}{2} |u(t, x)|^2.$$

Note that, as $u^*(t, \cdot)$ is a closed one-form on M , it holds that

$$\nabla |u|^2 = 2u \cdot \nabla u.$$

Hence we can check that

$$\partial_t (\nabla \phi - u) = -\gamma (\nabla \phi - u).$$

Note that at $t = 0$, $u(0) = \nabla \phi(0)$. Thus $u(t) = \nabla \phi(t)$ on $[0, T] \times M$. Substituting this into the compressible Euler equation with damping, we have

$$\nabla \left(\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 \right) = -\gamma \nabla \phi - \frac{1}{c^2} \nabla V'(\rho).$$

This proves that the Cauchy problem of the transport equation (11) and the deformed Hamilton-Jacobi equation (12) has a unique solution $(\rho, \phi) \in C([0, T], H^s(M) \times H^{s+1}(M))$ for all $s > \frac{n}{2} + 1$. The proof is completed. \square

7 Entropy dissipation formulas for Langevin deformation on Wasserstein space

In this section we prove the entropy dissipation formulas along the Langevin deformation of flows on the Wasserstein space. In this subsection, we assume that M is Euclidean space or a compact Riemannian manifolds. By Theorem 6.4, for any $c \in [0, \infty]$, the Langevin deformation of flows on $T^*P_2(M, \mu)$ has a unique smooth solution (up to an additional constant) on $[0, T_c] \times P_2^\infty(M, \mu)$.

7.1 Entropy formula for Langevin deformation

By Otto's infinite dimensional Riemannian metric on $T_{\rho d\mu} P_2^\infty(M, \mu)$, we have

$$\|\dot{\rho}\|^2 = \int_M |\nabla \phi|^2 \rho d\mu.$$

We now prove the following

Theorem 7.1 *Let (ϕ_t, ρ_t) be a smooth solution to the Langevin deformation of geometric flows (63) and (64) on $T^* P_2^\infty(M, \mu)$, i.e.,*

$$\begin{aligned} \partial_t \rho + \nabla_\mu^* (\rho \nabla \phi) &= 0, \\ c^2 \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 \right) &= -\phi + \frac{\delta V}{\delta \rho}. \end{aligned}$$

Then (ρ, ϕ) interpolates the geodesic flow (ρ, ϕ) , which satisfies (1) with (2), and the backward gradient flow (62) of V on $P_2^\infty(M, \mu)$, which satisfies

$$\partial_t \rho = \nabla_\mu^* \left(\rho \nabla \frac{\delta V}{\delta \rho} \right),$$

Let

$$H(\rho, v) = c^2 \int_M |\nabla \phi(x)|^2 \rho d\mu + V(\rho).$$

Then

$$\begin{aligned} \frac{d^2}{dt^2} H(\rho_t, v_t) &= 2 \left\| \frac{v}{c^2} + \frac{\nabla V(\rho)}{c} \right\|^2 + 2 \nabla^2 V(\rho) \left(\frac{v}{c}, \frac{v}{c} \right) \\ &= c^2 \|\dot{\rho}\|^2 + 2 \nabla^2 V \left(\frac{v}{c}, \frac{v}{c} \right). \end{aligned}$$

In particular, if V is K -convex, i.e., $\nabla^2 V \geq K$, then

$$\frac{d^2}{dt^2} H(\rho_t, v_t) \geq 2K \|\dot{\rho}\|^2 + 2\|\dot{v}\|^2.$$

Proof. Theorem 7.1 can be proved by straightforward extending the argument used in the proof of Theorem 5.1 to the Wasserstein space $P_2^\infty(M, \mu)$ equipped with Otto's infinite dimensional Riemannian metric. To save the length of the paper, we omit the detail here. \square

7.2 Proof of Theorem 1.5

Applying Theorem 7.1 to $V(\rho) = \text{Ent}(\rho) = \int_M \rho \log \rho d\mu$, we can derive Theorem 1.5.

Theorem 7.2 *Let $c > 0$. Let ρ, ϕ be a smooth solution of the following equations*

$$\begin{aligned} \partial_t \rho + \nabla_\mu^* (\rho \nabla \phi) &= 0, \\ c^2 \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 \right) &= -\phi + \log \rho + 1. \end{aligned}$$

Let

$$H(\rho, \phi) = \frac{c^2}{2} \int_M |\nabla \phi|^2 \rho d\mu + \text{Ent}(\rho).$$

Then

$$\begin{aligned} \frac{d}{dt} H(\rho) &= 2 \int_M \nabla \phi \cdot \nabla \rho d\mu - \int_M |\nabla \phi|^2 \rho d\mu \\ &= - \int_M (2L\phi + |\nabla \phi|^2) \rho d\mu, \end{aligned}$$

and

$$\frac{d^2}{dt^2} H(\rho, \phi) = 2 \int_M [c^{-2} |\nabla \phi - \nabla \log \rho|^2 + |\text{Hess} \phi|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi)] \rho d\mu. \quad (79)$$

Proof. Indeed, applying Theorem 7.1 to $V = \text{Ent}$, we have

$$\frac{d^2}{dt^2} H(\rho, \phi) = 2c^2 \|\ddot{\rho}\|^2 + 2\text{Hess}_{P_2^\infty(M)} \text{Ent}(\rho) (\dot{\rho}, \dot{\rho}).$$

Here $\text{Hess}_{P_2^\infty(M)} \text{Ent}$ is the Hessian of Ent on the Wasserstein space $P_2^\infty(M)$ equipped with Otto's infinite dimensional Riemannian metric. By Theorem 2.4, we have

$$\text{Hess}_{P_2^\infty(M)} \text{Ent}(\rho) (\dot{\rho}, \dot{\rho}) = \int_M (|\text{Hess} \phi|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi)) \rho d\mu.$$

On the other hand, we have

$$\begin{aligned} c^2 \|\ddot{\rho}\|^2 &= c^2 \int_M \left| \nabla \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 \right) \right|^2 \rho d\mu \\ &= c^{-2} \int_M |\nabla \phi - \nabla \log \rho|^2 \rho d\mu. \end{aligned}$$

This finishes the proof of Theorem 7.2. \square

On compact manifolds with non-negative Bakry-Emery Ricci curvature, Theorem 7.2 implies that the Hamiltonian function H is always convex along the deformed flow curve (ρ, ϕ) which interpolates the geodesic flow and the backward gradient flow of V on the Wasserstein space over a compact Riemannian manifold with weighted measure.

Note that, when $c = 0$ in Theorem 7.2, we have $\phi = \log \rho + 1$, hence $\nabla \phi - \rho^{-1} \nabla \rho = 0$. This yields $\partial_t \rho = -L\rho$, and

$$\frac{d^2}{dt^2} \text{Ent}(\rho(t)) = 2 \int_M [|\text{Hess} \phi|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi)] \rho d\mu.$$

If we formally take $c = \infty$ in Theorem 7.2, we have $\ddot{\rho} = 0$, and we obtain

$$\frac{d^2}{dt^2} H(\rho, \phi) = 2 \int_M [|\text{Hess} \phi|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi)] \rho d\mu.$$

However, this formula is not correct. Indeed, when $c = \infty$, the kinetic energy term $\frac{1}{2} \int_M |\nabla \phi(t)|^2 \rho(t) d\mu$ is a constant along the geodesic flow $(\rho(t), \phi(t))$ on the Wasserstein

space $P_2(M, \mu)$, thus $\frac{c^2}{2} \int_M |\nabla \phi(t)|^2 \rho(t) d\mu = \infty$. In this case, we must replace the left hand side of (79) in Theorem 7.2 by the second order derivative of $\text{Ent}(\rho(t))$, which is given by the entropy dissipation formula (Theorem 2.4).

$$\frac{d^2}{dt^2} \text{Ent}(\rho(t)) = \int_M [|\text{Hess} \phi|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi)] \rho d\mu.$$

In other words, we have the following

Corollary 7.3 *Let M be a compact Riemannian manifold, $f \in C^2(M)$. Then*
(i) When $c = \infty$, we have $\ddot{\rho} = 0$, i.e., (ρ, ϕ) is a geodesic flow on $P_2(M, \mu)$, and satisfies the transport equation (1) and the Hamilton-Jacobi equation (2). Moreover

$$\frac{d^2}{dt^2} \text{Ent}(\rho(t)) = \int_M [|\text{Hess} \phi|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi)] \rho d\mu,$$

(ii) When $c = 0$, we have $\phi = \log \rho + 1$, i.e., ρ is a positive solution to the backward heat equation

$$\partial_t \rho = -L\rho.$$

Moreover

$$\frac{d^2}{dt^2} \text{Ent}(\rho(t)) = 2 \int_M [|\text{Hess} \phi|^2 + \text{Ric}(L)(\nabla \phi, \nabla \phi)] \rho d\mu.$$

7.3 Langevin deformation on $T^*P_2^\infty(\mathbb{R}^m, dx)$

Let $m \in \mathbb{N}$, $m \geq n$. In this subsection we introduce the model of deformation of flows on $T^*P_2^\infty(\mathbb{R}^m)$. Let u be a positive solution of the following ODE on $[0, T) \subset [0, \infty)$

$$c^2 u'' + u' = -\frac{1}{2u}, \quad (80)$$

with given initial datas $u(0) > 0$ and $u'(0) \in \mathbb{R}$. Note that, for all $T > 0$, in the case $c = 0$, $u(t) = \sqrt{T-t}$ is a solution to (80) on $[0, T)$, and in the case $c = \infty$, $u(t) = T - t$ (or $u(t) = t$) is a solution to (80) on $[0, T)$ (or $(0, T]$).

The following result provides the reference model for the deformation of flows on $T^*P_2^\infty(\mathbb{R}^m)$.

Theorem 7.4 *Let u be a smooth solution to the ODE (80). Let $\alpha(t) = \frac{u'(t)}{u(t)}$, and $\beta(t)$ be a smooth function such that*

$$c^2 \dot{\beta}(t) = -\beta(t) - m \log u(t) - \frac{m}{2} \log(4\pi) + 1,$$

with a given initial data $\beta(0) \in \mathbb{R}$. For $x \in \mathbb{R}^m$ and $t > 0$, let

$$\begin{aligned} \phi(x, t) &= \frac{\alpha(t)}{2} \|x\|^2 + \beta(t), \\ \rho(x, t) &= \frac{1}{(4\pi u^2(t))^{m/2}} e^{-\frac{\|x\|^2}{4u^2(t)}}. \end{aligned}$$

Then $(\rho(x, t), \phi(x, t))$ satisfies the transport equation and the deformed Hamilton-Jacobi equation on \mathbb{R}^m

$$\partial_t \rho + \operatorname{div}(\rho \nabla \phi) = 0, \quad (81)$$

$$c^2 \left(\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 \right) = -\phi + \log \rho + 1. \quad (82)$$

Proof. Note that $\nabla \phi(x, t) = \alpha(t)x$. The transport equation (81) has a special solution given by

$$\rho(x, t) = \gamma^m(t) \rho_0(\gamma(t)x),$$

where $\rho_0(x)$ is any probability density on \mathbb{R}^m with respect to the Lebesgue measure, and γ is a smooth function in t which will be determined later. Indeed, we have

$$\begin{aligned} \partial_t \rho &= m\gamma^{m-1} \dot{\gamma} \rho_0(\gamma x) + \sum_{i=1}^m \gamma^m \dot{\gamma} x_i \partial_{x_i} \rho_0(\gamma x), \\ \operatorname{div}(\rho \nabla \phi) &= m\gamma^m \alpha \rho_0(\gamma x) + \gamma^m \alpha \langle \nabla \rho_0(\gamma x), x \rangle \\ &= m\gamma^m \alpha \rho_0(\gamma x) + \sum_{i=1}^m \gamma^{m+1} \alpha x_i \partial_{x_i} \rho_0(\gamma x). \end{aligned}$$

Thus, (ρ, ϕ) satisfies the transport equation if and only if

$$\dot{\gamma} + \gamma \alpha = 0.$$

Substituting ϕ and ρ into (82), we have

$$[c^2(\dot{\alpha} + \alpha^2) + \alpha] \frac{|x|^2}{2} + c^2 \dot{\beta} = -\beta(t) + m \log \gamma(t) + \log \rho_0(\gamma(t)x) + 1.$$

Changing the variable $y = \gamma(t)x$, we have

$$[c^2(\dot{\alpha}(t) + \alpha^2(t)) + \alpha(t)] \frac{|y|^2}{2\gamma^2(t)} + c^2 \dot{\beta}(t) = -\beta(t) + m \log \gamma(t) + \log \rho_0(y) + 1.$$

In particular, taking

$$\rho_0(y) = \frac{1}{(4\pi)^{\frac{m}{2}}} e^{-\frac{\|y\|^2}{4}}, \quad \forall y \in \mathbb{R}^m,$$

then we can choose $\alpha(t)$ and $\beta(t)$ by solving the following ODEs

$$\begin{aligned} c^2(\dot{\alpha}(t) + \alpha^2(t)) + \alpha(t) &= -\frac{\gamma^2(t)}{2}, \\ c^2 \dot{\beta}(t) &= -\beta(t) + m \log \gamma(t) - \frac{n}{2} \log(4\pi) + 1. \end{aligned}$$

Let $u(t) = e^{\int_0^t \alpha(s) ds}$, and assume $\gamma(0) = 1$. Then $\alpha = \frac{u'}{u}$, $\dot{\alpha} = \frac{u''}{u} - \frac{u'^2}{u^2}$, $\gamma(t) = \frac{1}{u(t)}$, and

$$c^2 \left(\frac{u''}{u} - \frac{u'^2}{u^2} + \frac{u'^2}{u^2} \right) + \frac{u'}{u} = -\frac{1}{2u^2}.$$

Thus u satisfies the following nonlinear ODE

$$c^2 u'' + u' = -\frac{1}{2u}.$$

In particular, for any $T > 0$, when $c = 0$, we can take $u(t) = \sqrt{T-t}$, $\alpha(t) = -\frac{1}{2(T-t)}$, $\beta(t) = -\frac{m}{2} \log(4\pi(T-t)) + 1$ and $\gamma(t) = \frac{1}{\sqrt{T-t}}$, $t \in [0, T)$; and when $c = \infty$, we can take $u(t) = T-t$, $\alpha(t) = -\frac{1}{T-t}$, $\beta(t) = 0$, and $\gamma(t) = \frac{1}{T-t}$, $t \in [0, T)$, or $u(t) = t$, $\alpha(t) = \frac{1}{t}$, $\beta(t) = 0$, and $\gamma(t) = \frac{1}{t}$, $t \in (0, T]$.

Remark 7.5 The equation (80) has the following interpretation: Let V be a smooth function on $\mathbb{R}^+ \setminus \{0\} = (0, \infty)$. Consider the following Newton-Langevin equation on $T^*\mathbb{R}^+ \setminus \{0\} = (0, \infty) \times \mathbb{R}$

$$\begin{aligned} \dot{x} &= \frac{v}{c} \\ \dot{v}_t &= -\frac{v}{c^2} + \frac{\nabla V(x)}{c}. \end{aligned}$$

Then x satisfies the Langevin equation

$$c^2 \ddot{x} = -\dot{x} + \nabla V(x).$$

In particular, taking

$$V(x) = -\frac{1}{2} \log x, \quad x > 0,$$

which is Lipschitz on $[\delta, +\infty)$ for any $\delta > 0$, we have

$$\nabla V(x) = -\frac{1}{2x}.$$

Thus

$$c^2 \ddot{x} + \dot{x} = -\frac{1}{2x}$$

can be realized as the solution of the Newton-Langevin equation on $T^*\mathbb{R}^+ \setminus \{0\} = (0, \infty) \times \mathbb{R}$. Given any initial position $x(0) > 0$ and initial velocity $x'(0) \in \mathbb{R}$, there exists a unique solution $x(t)$ to the above equations on a small interval $[0, T) \subset [0, \infty)$ with the given initial data $x(0)$ and $x'(0)$.

Remark 7.6 Let $v(t) = u(T-t)$. Then $v'(t) = -u'(T-t)$, and $v''(t) = u''(T-t)$. Moreover, v satisfies the following nonlinear ODE on the positive real line

$$c^2 v'' - v' = -\frac{1}{2v}.$$

In particular, when $c = 0$, we can take $v(t) = \sqrt{t}$, and when $c = \infty$, we can take $v(t) = t$.

Proposition 7.7 Let (ρ_m, ϕ_m) be the special solution of (81) and (82) in Theorem 7.4. Then

$$\begin{aligned} \text{Ent}(\rho_m(t)) &= -\frac{m}{2} (1 + \log(4\pi u^2(t))), \\ H(\rho_m(t), \phi_m(t)) &= \frac{mc^2 u'(t)^2}{2} - \frac{m}{2} (1 + \log(4\pi u^2(t))). \end{aligned}$$

Proof. Indeed, $\rho_m(t)$ Theorem 7.4 is the Gaussian heat kernel at time $u^2(t)$. □

8 W -entropy formula for Langevin deformation on Wasserstein space

In this section we prove Perelman's type W -entropy formulas and monotonicity results for the deformed flows on the Wasserstein space over compact Riemannian manifolds or on Euclidean spaces. Our results can be regarded as an interpolation between the W -entropy formula for the geodesic flow and the the heat flow on the Wasserstein space over compact Riemannian manifolds. We also provide the rigidity models for the W -entropy of the deformed flows on the Wasserstein space over complete noncompact Riemannian manifolds.

8.1 W -entropy formula for Langevin deformation under $CD(0, \infty)$ -condition

The following result extend the W -entropy formulas in Theorem 5.3 to the deformed flow $(\rho(t), \phi(t))$ to $T^*P_2^\infty(M, \mu)$.

Theorem 8.1 *Under the same notation as in Theorem 7.2, for any $c > 0$, we have*

$$\begin{aligned} \left(\frac{d^2}{dt^2} + \frac{1}{c^2} \frac{d}{dt} \right) \text{Ent}(\rho(t)) &= \frac{1}{c^2} \int_M \frac{|\nabla \rho|^2}{\rho} d\mu + \int_M [|\text{Hess}\phi|^2 + \text{Ric}(L)(\nabla\phi, \nabla\phi)] \rho d\mu, \\ \left(\frac{d^2}{dt^2} + \frac{2}{c^2} \frac{d}{dt} \right) H(\rho(t), \phi(t)) &= \frac{2}{c^2} \int_M \frac{|\nabla \rho|^2}{\rho} d\mu + \int_M [|\text{Hess}\phi|^2 + \text{Ric}(L)(\nabla\phi, \nabla\phi)] \rho d\mu. \end{aligned}$$

Let

$$\begin{aligned} W_{H,c}(\rho(t), \phi(t)) &= H(\rho(t), \phi(t)) + \frac{c^2(1 - e^{\frac{2t}{c^2}})}{2} \frac{d}{dt} H(\rho(t), \phi(t)), \\ W_{\text{Ent},c}(\rho(t)) &= \text{Ent}(\rho(t)) + c^2(1 - e^{\frac{t}{c^2}}) \frac{d}{dt} \text{Ent}(\rho(t)). \end{aligned}$$

Then

$$\begin{aligned} \frac{d}{dt} W_{H,c}(\rho(t), \phi(t)) &= (1 - e^{\frac{2t}{c^2}}) \int_M \frac{|\nabla \rho|^2}{\rho} d\mu + \int_M [|\text{Hess}\phi|^2 + \text{Ric}(L)(\nabla\phi, \nabla\phi)] \rho d\mu, \\ \frac{d}{dt} W_{\text{Ent},c}(\rho(t)) &= (1 - e^{\frac{t}{c^2}}) \int_M \frac{|\nabla \rho|^2}{\rho} d\mu + \int_M [|\text{Hess}\phi|^2 + \text{Ric}(L)(\nabla\phi, \nabla\phi)] \rho d\mu. \end{aligned}$$

In particular, if $\text{Ric}(L) \geq 0$, then for all $c > 0$, we have

$$\frac{d}{dt} W_{H,c}(\rho(t), \phi(t)) \leq 0, \quad \forall t \geq 0,$$

and

$$\frac{d}{dt} W_{\text{Ent},c}(\rho(t)) \leq 0, \quad \forall t \geq 0.$$

Proof. By Theorem 7.2, we can prove Theorem 8.1 similarly to the one of Theorem 5.3. \square

8.2 W -entropy formula for Langevin deformation under $CD(0, m)$ -condition

We are in a position to prove Theorem 1.6. More precisely, we prove the following

Theorem 8.2 *Let $c \in [0, \infty)$, and let $(\rho(t), \phi(t))$ be the deformed flow. Let $\alpha(t) = (\log u)'$. In the case $m = n$, and $L = \Delta$, we have*

$$\begin{aligned} & \frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \left(2\alpha(t) + \frac{1}{c^2} \right) \frac{d}{dt} \text{Ent}(\rho(t)) + n\alpha^2(t) \\ &= \int_M \left[|\text{Hess}\phi - \alpha(t)g|^2 + \text{Ric}(\nabla\phi, \nabla\phi) \right] \rho dv + \frac{1}{c^2} \int_M \frac{|\nabla\rho|^2}{\rho} dv. \end{aligned}$$

In the case $m > n$, we have

$$\begin{aligned} & \frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \left(2\alpha(t) + \frac{1}{c^2} \right) \frac{d}{dt} \text{Ent}(\rho(t)) + m\alpha^2(t) \\ &= \int_M \left[|\text{Hess}\phi - \alpha(t)g|^2 + \text{Ric}_{m,n}(L)(\nabla\phi, \nabla\phi) \right] \rho d\mu \\ & \quad + (m-n) \int_M \left| \alpha(t) + \frac{\nabla\phi \cdot \nabla f}{m-n} \right|^2 \rho d\mu + \frac{1}{c^2} \int_M \frac{|\nabla\rho|^2}{\rho} d\mu. \end{aligned}$$

Proof. The proof has the same spirit as those for Theorem 1.1 and Theorem 1.3. By Otto's calculation on $P_2(M, \mu)$, we have

$$\begin{aligned} \frac{d}{dt} \text{Ent}(\rho(t)) &= \nabla \text{Ent}(\rho(t)) \cdot \dot{\rho}(t) = \int_M \nabla(\log \rho + 1) \cdot \nabla \phi \rho d\mu \\ &= \int_M \nabla \phi \cdot \nabla \rho d\mu = - \int_M L\phi \rho d\mu, \end{aligned}$$

and by Theorem 2.4, we have

$$\begin{aligned} \frac{d^2}{dt^2} \text{Ent}(\rho(t)) &= \nabla^2 \text{Ent}(\rho(t))(\dot{\rho}(t), \dot{\rho}(t)) + \nabla \text{Ent} \cdot \ddot{\rho}(t) \\ &= \int_M (|\text{Hess}\phi|^2 + \text{Ric}(L)(\nabla\phi, \nabla\phi)) \rho d\mu + \int_M \nabla(\log \rho + 1) \cdot \nabla \left(\partial_t \phi + \frac{1}{2} |\nabla\phi|^2 \right) \rho d\mu \\ &= \int_M (|\text{Hess}\phi|^2 + \text{Ric}(L)(\nabla\phi, \nabla\phi)) \rho d\mu + \frac{1}{c^2} \int_M \nabla \rho \cdot \nabla(-\phi + \log \rho + 1) d\mu \\ &= \int_M (|\text{Hess}\phi|^2 + \text{Ric}(L)(\nabla\phi, \nabla\phi)) \rho d\mu + \frac{1}{c^2} \int_M \frac{|\nabla\rho|^2}{\rho} d\mu + \frac{1}{c^2} \int_M L\phi \rho d\mu. \end{aligned}$$

In the case $m = n$, $L = \Delta$ and $\mu = v$, we have

$$\begin{aligned} & \frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \left(2\alpha(t) + \frac{1}{c^2} \right) \frac{d}{dt} \text{Ent}(\rho(t)) + n\alpha^2(t) \\ &= \int_M [|\text{Hess}\phi|^2 + \text{Ric}(\nabla\phi, \nabla\phi)] \rho d\mu + \frac{1}{c^2} \int_M \frac{|\nabla\rho|^2}{\rho} dv - 2\alpha \int_M \Delta\phi \rho d\mu + n\alpha^2 \\ &= \int_M |\text{Hess}\phi - \alpha(t)g|^2 \rho d\mu + \int_M \text{Ric}(\nabla\phi, \nabla\phi) \rho d\mu + \frac{1}{c^2} \int_M \frac{|\nabla\rho|^2}{\rho} dv. \end{aligned}$$

In general case $m > n$, note that

$$\begin{aligned}
& Ric(L)(\nabla\phi, \nabla\phi) + 2\nabla\phi \cdot \nabla f + (m-n)\alpha^2 \\
&= Ric_{m,n}(\nabla\phi, \nabla\phi) + \frac{|\nabla\phi \cdot \nabla f|^2}{m-n} + 2\nabla\phi \cdot \nabla f + (m-n)\alpha^2 \\
&= Ric_{m,n}(\nabla\phi, \nabla\phi) + (m-n) \left| \alpha + \frac{\nabla\phi \cdot \nabla f}{m-n} \right|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
& \frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \left(2\alpha(t) + \frac{1}{c^2} \right) \frac{d}{dt} \text{Ent}(\rho(t)) + n\alpha^2(t) \\
&= \int_M [|\text{Hess}\phi|^2 + Ric(L)(\nabla\phi, \nabla\phi)]\rho + \frac{1}{c^2} \int_M \frac{|\nabla\rho|^2}{\rho} d\mu - 2\alpha(t) \int_M L\phi\rho d\mu + m\alpha^2(t) \\
&= \int_M |\text{Hess}\phi - \alpha(t)g|^2 \rho d\mu + 2\alpha(t) \int_M \Delta\phi\rho d\mu + \int_M Ric(L)(\nabla\phi, \nabla\phi)\rho d\mu \\
&\quad + \frac{1}{c^2} \int_M \frac{|\nabla\rho|^2}{\rho} d\mu - 2\alpha \int_M L\phi\rho d\mu + (m-n)\alpha^2(t) \\
&= \int_M |\text{Hess}\phi - \alpha(t)g|^2 \rho d\mu + \int_M Ric(L)(\nabla\phi, \nabla\phi)\rho d\mu \\
&\quad + \frac{1}{c^2} \int_M \frac{|\nabla\rho|^2}{\rho} d\mu + 2\alpha \int_M \nabla f \cdot \nabla\phi\rho d\mu + (m-n)\alpha^2(t) \\
&= \int_M \left[|\text{Hess}\phi - \alpha(t)g|^2 + Ric_{m,n}(L)(\nabla\phi, \nabla\phi) \right] \rho d\mu \\
&\quad + (m-n) \int_M \left| \alpha(t) + \frac{\nabla\phi \cdot \nabla f}{m-n} \right|^2 \rho d\mu + \frac{1}{c^2} \int_M \frac{|\nabla\rho|^2}{\rho} d\mu.
\end{aligned}$$

This completes the proof of Theorem 1.6. \square

8.3 Rigidity theorem for W -entropy formula for Langevin deformation

The following result says that the entropy inequality in Theorem 1.6 is an equality if $(M, g, f) = (\mathbb{R}^m, g_0, 0)$ and $(\rho, \phi) = (\rho_m, \phi_m)$ on $T^*P_2^\infty(\mathbb{R}^m, dx)$. This provides us the rigidity model for the entropy formula on the Langevin deformation of flows on complete Riemannian manifolds with $CD(0, m)$ condition and with bounded geometry condition and natural growth condition on (ρ, ϕ) on $T^*P_2^\infty(M, \mu)$.

Theorem 8.3 *Let $\alpha(t) = \frac{u'(t)}{u(t)}$. For the model (ρ_m, ϕ_m) on (\mathbb{R}^m, dx) , we have*

$$- n\alpha^2(t) = \frac{d^2}{dt^2} \text{Ent}(\rho_m) + \left(2\alpha(t) + \frac{1}{c^2} \right) \frac{d}{dt} \text{Ent}(\rho_m) - \frac{1}{c^2} \int_{\mathbb{R}^m} \frac{|\nabla\rho_m|^2}{\rho_m} dx. \quad (83)$$

Proof. For the model (ρ_m, ϕ_m) on (\mathbb{R}^m, dx) , we have

$$\begin{aligned} \text{Hess}\phi_m &= \alpha(t)g, \\ \Delta\phi_m &= n\alpha(t), \\ \nabla \log \rho_m &= -\frac{x}{2u^2(t)}, \\ \text{Ric}_{m,n}(L) &= 0. \end{aligned}$$

Thus, (83) follows from Theorem 8.2. \square

8.4 Comparison of W -entropy for Langevin deformation

Let us define the W -entropy for the Langevin deformation of flows on the cotangent bundle over the Wasserstein space $P_2^\infty(M, \mu)$. Its derivative satisfies

$$\frac{d}{dt}W(\rho(t)) = \frac{d^2}{dt^2}\text{Ent}(\rho(t)) + \left(2\alpha(t) + \frac{1}{c^2}\right) \frac{d}{dt}\text{Ent}(\rho(t)) - \frac{1}{c^2} \int_M \frac{|\nabla \rho|^2}{\rho} d\mu.$$

The corresponding W -entropy on the model (ρ_m, ϕ_m) on (\mathbb{R}^m, dx) satisfies

$$\begin{aligned} \frac{d}{dt}W(\rho_m(t)) &= \frac{d^2}{dt^2}\text{Ent}(\rho_m(t)) + \left(2\alpha(t) + \frac{1}{c^2}\right) \frac{d}{dt}\text{Ent}(\rho_m(t)) - \frac{1}{c^2} \int_{\mathbb{R}^n} \frac{|\nabla \rho_m|^2}{\rho_m} dx \\ &= -n\alpha^2(t). \end{aligned}$$

In view of this, Theorem 8.2 can be reformulated as follows.

Theorem 8.4 *Let $\alpha(t) = \frac{u'(t)}{u(t)}$. Define the W -entropy as follows*

$$W(\rho(t)) = \frac{d}{dt}\text{Ent}(\rho(t)) + \int_0^t \left(2\alpha(s) + \frac{1}{c^2}\right) \frac{d}{ds}\text{Ent}(\rho(s)) ds - \frac{1}{c^2} \int_0^t \int_M \frac{|\nabla \rho|^2}{\rho} d\mu ds,$$

Then

$$\frac{d}{dt}W(\rho(t)) = \frac{d^2}{dt^2}\text{Ent}(\rho(t)) + \left(2\alpha(t) + \frac{1}{c^2}\right) \frac{d}{dt}\text{Ent}(\rho(t)) - \frac{1}{c^2} \int_M \frac{|\nabla \rho|^2}{\rho} d\mu.$$

Moreover

$$\begin{aligned} \frac{d}{dt}(W(\rho(t)) - W(\rho_m(t))) &= \int_M |\text{Hess}\phi - \alpha(t)g|^2 \rho dv + \int_M \text{Ric}_{m,n}(L)(\nabla\phi, \nabla\phi) \rho d\mu \\ &\quad + \frac{1}{m-n} \int_M |\nabla f \cdot \nabla\phi + (m-n)\alpha(t)|^2 \rho d\mu. \end{aligned}$$

In particular, if $\text{Ric}_{m,n}(L) \geq 0$, then for all $t > 0$, we have the comparison theorem

$$\frac{d}{dt}W(\rho(t)) \geq \frac{d}{dt}W(\rho_m(t)) \tag{84}$$

Remark 8.5 Theorem 1.1 can be derived from Theorem 1.6 (i.e., Theorem 8.2). Indeed, when $c = \infty$, we have $u(t) = t$, $\alpha(t) = \frac{1}{t}$, the derivative of W -entropy is given by

$$\frac{d}{dt}W(\rho(t)) = \frac{d^2}{dt^2}\text{Ent}(\rho(t)) + \frac{2}{t}\text{Ent}(\rho(t)),$$

and the W -entropy formula reads as (i.e., Theorem 1.1)

$$\begin{aligned} \frac{d}{dt}(W(\rho(t)) - W(\rho_m(t))) &= \int_M \left| \text{Hess}\phi - \frac{g}{t} \right|^2 \rho d\mu + \int_M \text{Ric}_{m,n}(L)(\nabla\phi, \nabla\phi) \rho d\mu \\ &\quad + \frac{1}{m-n} \int_M \left| \nabla f \cdot \nabla\phi + \frac{m-n}{t} \right|^2 \rho d\mu. \end{aligned}$$

When $c = 0$, we have $\phi = \log \rho + 1$, $u(t) = \sqrt{T-t} = \sqrt{\tau}$, $\alpha(t) = \frac{1}{2(t-T)} = -\frac{1}{2\tau}$, $\tau = T - t$, and

$$\frac{d}{d\tau}\text{Ent}(\rho(\tau)) = - \int_M \frac{|\nabla\rho|^2}{\rho} d\mu.$$

In this case, we can prove that the derivative of W -entropy is also given by

$$\frac{d}{d\tau}W(\rho(\tau)) = \frac{d^2}{d\tau^2}\text{Ent}(\rho(\tau)) + \frac{2}{\tau} \frac{d}{d\tau}\text{Ent}(\rho(\tau)),$$

and the W -entropy formula reads as (i.e., Theorem 1.3)

$$\begin{aligned} \frac{d}{d\tau}(W(\rho(\tau)) - W(\rho_m(\tau))) &= 2 \int_M \left| \text{Hess} \log \rho + \frac{g}{2\tau} \right|^2 \rho d\mu + 2 \int_M \text{Ric}_{m,n}(L)(\nabla \log \rho, \nabla \log \rho) \rho d\mu \\ &\quad + \frac{2}{m-n} \int_M \left| \nabla f \cdot \nabla \log \rho - \frac{m-n}{2\tau} \right|^2 \rho d\mu. \end{aligned}$$

Note that the coefficient in the right hand side of the above W -entropy formulas is 1 for $c = \infty$ but is 2 for $c = 0$. For its reason, see Section 3.1 and Section 3.2.

9 W -entropy inequalities under entropic curvature-dimension condition

In [9], Erbar-Kuwada-Sturm introduced a new definition of the curvature-dimension condition on metric-measure spaces, called the entropic curvature-dimension condition. By [9], we say that the entropic curvature-dimension condition, denoted by $CD_{\text{Ent}}(K, N)$, holds if the Boltzmann entropy Ent satisfies

$$\text{HessEnt} - \frac{1}{N} \nabla \text{Ent}^{\otimes 2} \geq K,$$

where $K \in \mathbb{R}$, $N \geq n$ are two constants. As was pointed out in [9], when M is a complete Riemannian manifold, the $CD_{\text{Ent}}(K, N)$ is equivalent to the $CD(K, N)$ -condition.

The purpose of this section is to prove the W -entropy inequalities for the geodesic flow, the gradient flows as well as the Langevin deformation of flows on the Wasserstein space over complete Riemannian manifolds with Erbar-Kawada-Sturm's $CD_{\text{Ent}}(K, N)$ condition. This might bring some new insights to the study of geometric analysis on non smooth metric measure spaces.

9.1 W -entropy inequalities for geodesic and gradient flows

In this section, we show that, under Erbar-Kawada-Sturm's $CD_{\text{Ent}}(K, N)$ -condition, we have the following result.

Theorem 9.1 *Let M be a complete Riemannian manifold of dimension n . Suppose that Erbar-Kawada-Sturm's $CD_{\text{Ent}}(K, N)$ condition holds, i.e.,*

$$\text{HessEnt} - \frac{1}{N} \nabla \text{Ent}^{\otimes 2} \geq K,$$

where $K \in \mathbb{R}$, $N \geq n$ are two constants. Then

(i) *for geodesic flow $(\rho(t), \phi(t))$ on $T^*P_2(M, \mu)$, we have*

$$\frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \frac{2}{t} \frac{d}{dt} \text{Ent}(\rho(t)) + \frac{N}{t^2} \geq \frac{1}{N} \left| \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle \right|^2 + K |\dot{\rho}(t)|^2.$$

(ii) *for the backward gradient flow $\dot{\rho}(t) = \nabla \text{Ent}(\rho(t))$ on $P_2(M, \mu)$, we have⁷*

$$\frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \frac{2}{t} \frac{d}{dt} \text{Ent}(\rho(t)) + \frac{N}{2} \left(K + \frac{1}{t} \right)^2 \geq \frac{2}{N} \left| \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle \right|^2 + \frac{N}{2} \left(K + \frac{1}{t} \right)^2 g^2.$$

Proof. By Section 4.1, for geodesic flow $(\rho(t), \phi(t))$ on $T^*P_2(M, \mu)$, we have

$$\begin{aligned} \frac{d}{dt} \text{Ent}(\rho(t)) &= \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle, \\ \frac{d^2}{dt^2} \text{Ent}(\rho(t)) &= \text{HessEnt}((\dot{\rho}(t), \dot{\rho}(t))). \end{aligned}$$

Thus

$$\begin{aligned} \frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \frac{2}{t} \frac{d}{dt} \text{Ent}(\rho(t)) + \frac{N}{t^2} &= \text{HessEnt}(\rho(t)) + \frac{2}{t} \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle + \frac{N}{t^2} \\ &\geq \frac{1}{N} \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle^2 + K |\dot{\rho}(t)|^2 + \frac{2}{t} \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle + \frac{N}{t^2} \\ &= \frac{1}{N} \left| \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle \right|^2 + \frac{N}{t}^2 + K |\dot{\rho}(t)|^2. \end{aligned}$$

On the other hand, for the backward gradient flow $\dot{\rho}(t) = \nabla \text{Ent}(\rho(t))$ on $P_2(M, \mu)$, we have (see Section 3.1)

$$\begin{aligned} \frac{d}{dt} \text{Ent}(\rho(t)) &= |\nabla \text{Ent}(\rho(t))|^2 = |\dot{\rho}(t)|^2 = \int_M \frac{|\nabla \rho|^2}{\rho} d\mu, \\ \frac{d^2}{dt^2} \text{Ent}(\rho(t)) &= 2 \text{HessEnt}((\dot{\rho}(t), \dot{\rho}(t))). \end{aligned}$$

⁷Note that $\langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle = \int_M |\nabla \log \rho|^2 \rho d\mu$. Compare to (29) in Remark 3.3.

Thus

$$\begin{aligned}
& \frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \frac{2}{t} \frac{d}{dt} \text{Ent}(\rho(t)) + \frac{N}{2} \left(K + \frac{1}{t} \right)^2 \\
&= 2 \text{HessEnt}(\rho(t)) + \frac{2}{t} \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle + \frac{N}{2} \left(K + \frac{1}{t} \right)^2 \\
&\geq \frac{2}{N} \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle^2 + 2K |\dot{\rho}(t)|^2 + \frac{2}{t} \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle + \frac{N}{2} \left(K + \frac{1}{t} \right)^2 \\
&= \frac{2}{N} \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle^2 + 2 \left(K + \frac{1}{t} \right) \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle + \frac{N}{2} \left(K + \frac{1}{t} \right)^2 \\
&= \frac{2}{N} \left| \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle + \frac{N}{2} \left(K + \frac{1}{t} \right) g \right|^2.
\end{aligned}$$

This finishes the proof. \square

9.2 W -entropy inequality for Langevin deformation

Theorem 9.2 *Let $c \in [0, \infty)$, and let $(\rho(t), \phi(t))$ be the Langevin deformation of flows on $T^*P_2(M, \mu)$. Suppose that Erbar-Kawada-Sturm's $CD_{\text{Ent}}(K, N)$ -condition holds for some constants $K \in \mathbb{R}$ and $N \in \mathbb{N}$ with $N \geq n$, i.e.,*

$$\text{HessEnt} - \frac{1}{N} \nabla \text{Ent}^{\otimes 2} \geq K.$$

Let $\alpha(t) = (\log u)'$ be as in Section 6.4 with $m = N$. Then

$$\begin{aligned}
& \frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \left(2\alpha(t) + \frac{1}{c^2} \right) \frac{d}{dt} \text{Ent}(\rho(t)) + N\alpha^2(t) + \frac{1}{c^2} |\nabla \text{Ent}(\rho(t))|^2 \\
& \geq \frac{1}{N} |\langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle + N\alpha(t)|^2 + K |\dot{\rho}(t)|^2.
\end{aligned}$$

Proof. For the Langevin deformation of flows on $T^*P_2(M, \mu)$, we have

$$\begin{aligned}
\frac{d}{dt} \text{Ent}(\rho(t)) &= \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle, \\
\frac{d^2}{dt^2} \text{Ent}(\rho(t)) &= \text{HessEnt}(\rho(t))(\langle \dot{\rho}(t), \dot{\rho}(t) \rangle) + \langle \nabla \text{Ent}(\rho), \ddot{\rho} \rangle \\
&= \text{HessEnt}(\rho(t))(\langle \dot{\rho}(t), \dot{\rho}(t) \rangle) - \frac{1}{c^2} \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) + \nabla V(\rho) \rangle.
\end{aligned}$$

Under Erbar-Kawada-Sturm's $CD_{\text{Ent}}(K, N)$ -condition, we have

$$\begin{aligned}
& \frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \left(2\alpha(t) + \frac{1}{c^2}\right) \frac{d}{dt} \text{Ent}(\rho(t)) + N\alpha^2(t) \\
= & \text{HessEnt}(\rho(t))(\dot{\rho}(t), \dot{\rho}(t)) - \frac{1}{c^2} \langle \nabla \text{Ent}, \dot{\rho}(t) + \nabla V(\rho) \rangle \\
& + \left(2\alpha(t) + \frac{1}{c^2}\right) \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle + N\alpha^2(t) \\
\geq & \frac{1}{N} \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle^2 + K|\dot{\rho}(t)|^2 - \frac{1}{c^2} \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) + \nabla V(\rho) \rangle \\
& + \left(2\alpha(t) + \frac{1}{c^2}\right) \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle + N\alpha^2(t).
\end{aligned}$$

Now $V(\rho) = \text{Ent}(\rho) = \int_M \rho \log \rho d\mu$. We have

$$\langle \nabla \text{Ent}(\rho(t)), \nabla V(\rho(t)) \rangle = \langle \nabla \text{Ent}(\rho(t)), \nabla \text{Ent}(\rho(t)) \rangle = |\nabla \text{Ent}(\rho(t))|^2 = \int_M \frac{|\nabla \rho|^2}{\rho} d\mu.$$

Hence

$$\begin{aligned}
& \frac{d^2}{dt^2} \text{Ent}(\rho(t)) + \left(2\alpha(t) + \frac{1}{c^2}\right) \frac{d}{dt} \text{Ent}(\rho(t)) + N\alpha^2(t) \\
\geq & \frac{1}{N} \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle^2 + K|\dot{\rho}(t)|^2 - \frac{1}{c^2} \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle - \frac{1}{c^2} |\nabla \text{Ent}(\rho(t))|^2 \\
& + \left(2\alpha(t) + \frac{1}{c^2}\right) \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle + N\alpha^2(t) \\
= & \frac{1}{N} \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle^2 + K|\dot{\rho}(t)|^2 + 2\alpha(t) \langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle + N\alpha^2(t) - \frac{1}{c^2} |\nabla \text{Ent}(\rho(t))|^2 \\
= & \frac{1}{N} |\langle \nabla \text{Ent}(\rho(t)), \dot{\rho}(t) \rangle + N\alpha(t)|^2 - \frac{1}{c^2} |\nabla \text{Ent}(\rho(t))|^2 + K|\dot{\rho}(t)|^2.
\end{aligned}$$

This finishes the proof. \square

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Songzi Li

SCHOOL OF MATHEMATICAL SCIENCES, BEIJING NORMAL UNIVERSITY, NO. 19,
XINJIEKOUWAI ST., BEIJING 100875, CHINA

SCHOOL OF MATHEMATICAL SCIENCE, FUDAN UNIVERSITY, 220, HANDAN ROAD,
SHANGHAI, 200432, CHINA

and

INSTITUT DE MATHÉMATIQUES, UNIVERSITÉ PAUL SABATIER, 118, ROUTE DE
NARBONNE, 31062, TOULOUSE CEDEX 9, FRANCE

Xiang-Dong Li

ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE, CHINESE ACADEMY OF SCIENCES,
55, ZHONGGUANCUN EAST ROAD, BEIJING, 100190, CHINA
E-mail: xdli@amt.ac.cn